GOOD MEASURES ON CANTOR SPACE

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ABSTRACT. While there is, up to homeomorphism, only one Cantor space, i.e. one zero-dimensional, perfect, compact, nonempty metric space, there are many measures on Cantor space which are not topologically equivalent. The *clopen values set* for a full, nonatomic measure μ is the countable dense subset $\{\mu(U) : U$ is clopen$\}$ of the unit interval. It is a topological invariant for the measure. For the class of good measures it is a complete invariant. A full, nonatomic measure μ is *good* if whenever U, V are clopen sets with $\mu(U) < \mu(V)$, there exists W a clopen subset of V such that $\mu(W) = \mu(U)$. These measures have interesting dynamical properties. They are exactly the measures which arise from uniquely ergodic minimal systems on Cantor space. For some of them there is a unique generic measure-preserving homeomorphism. That is, within the Polish group of such homeomorphisms there is a dense, G_δ conjugacy class.

INTRODUCTION

A *Cantor space* is a nonempty, compact, perfect, zero-dimensional metric space. An *ordered Cantor space* is a Cantor space equipped with a closed total order so that the order topology agrees with the metric topology. Any two Cantor spaces are homeomorphic and any two ordered Cantor spaces are order isomorphic.

A Borel probability measure μ on a compact metric space X is called *full* when nonempty open subsets have positive measure and *nonatomic* when countable subsets have measure zero. There exist many homeomorphically distinct, full, nonatomic measures on Cantor space. In fact the *clopen values set* $S(\mu)$, defined to be the set of values of the measure on the clopen subsets of X, provides a topological invariant, although it is not a complete invariant. On the other hand, if X is ordered, then the *special clopen values set* $\tilde{S}(\mu)$ is defined to be the set of values of the measure on those clopen intervals in X which contain the minimum point m of X. $\tilde{S}(\mu)$ is a complete invariant with respect to a measure preserving, order isomorphism. These results are proved in Akin [1999] and are reviewed in Section 1 below. The same set, which they called the *gap invariant*, is introduced by Cooper and Pignataro [1988] for a related purpose.

A full, nonatomic measure μ on a Cantor space X is said to satisfy the *Subset Condition* if for all clopen subsets U and V of X, $\mu(U) \leq \mu(V)$ implies that there exists a clopen subset U_1 of V such that $\mu(U) = \mu(U_1)$. μ is called *good* when it

satisfies the Subset Condition. On the other hand, a measure μ and an order $\leq$ are *adapted* if $S(\mu) = \tilde{S}(\mu)$.

Theorem 0.1. *If a measure μ is adapted to the order of an ordered Cantor space X, then it is a good measure on X. Conversely, if μ is a good measure on a Cantor space X, then there exists an order on X which is adapted to μ.*

Together with its applications, this is the main result of Section 2. From this theorem it follows that the clopen values set is a complete invariant for good measures. If μ is good on X, then for any nonempty clopen subset V the relative measure μ_V is good on V. It then follows that if U and V are clopen subsets with the same measure, then there exists a measure preserving automorphism of X mapping U to V. From a sharpening of the theorem one can prove that if μ is good, then the group of μ preserving automorphisms acts transitively on X.

A subset S of the unit interval I which contains $0, 1$ is called *group-like* if $S = G \cap I$ with G an additive subgroup of $\mathbb{R}$, or, equivalently, if $S + \mathbb{Z}$ is a group. If $S + \mathbb{Z}$ is a rational vector space (or a subfield of $\mathbb{R}$), then we call S $\mathbb{Q}$-*like* (resp. *field-like*).

If μ is good, then $S(\mu)$ is group-like and μ is adapted to an order iff $\tilde{S}(\mu)$ is group-like. The other conditions are associated with stronger homogeneity conditions on μ. In Section 3 we prove that goodness of μ with $S(\mu)$ field-like is equivalent to the condition that for every nonempty clopen subset V of X the relative measure μ_V is homeomorphic to μ on X, together with the technical condition that $\alpha \in S(\mu)$ with $\alpha < 1/2$ implies $2\alpha \in S(\mu)$. I suspect that this conditon, a weakening of the assumption that $S(\mu)$ is group-like, is redundant, but I have been unable to eliminate it.

In Section 4 we consider automorphisms of good measures. Assume μ is a good measure on X. If $S(\mu) \cap \mathbb{Q}$ is infinite, then there is a μ preserving automorphism which is uniquely ergodic and which has an adding machine translation as an almost one-to-one factor. On the other hand, if α is an irrational in $S(\mu)$, then there is a μ preserving automorphism which is uniquely ergodic and which has translation by α on the circle $\mathbb{R}/\mathbb{Z}$ as an almost one-to-one factor. One of these two cases always applies and so every good measure is the invariant measure for some uniquely ergodic transformation on Cantor space. Glasner and Weiss [1995] prove the converse, that is, the invariant measure for a uniquely ergodic transformation on Cantor space is good. Together these results relate this little theory to the powerful machinery due to Giordano, Putnam and Skau [1995]. Indeed, in his review of recent results in topological dynamics [2002], Glasner shows how to derive several of our results from theirs, in particular avoiding our use of the order structure. On the other hand, the order structure allows us to make very explicit constructions.

The final result of Section 4 concerns the Rohlin Property. A Polish topological group G is said to satisfy the *Rohlin Property* if the adjoint action of G on G is topologically transitive, i.e. $TRANS(G) =_{def} \{x \in G : ad(G)x$ is dense in $G\}$ is nonempty, in which case, $TRANS(G)$ is a dense G_δ set. A compact metric space X is said to satisfy the Rohlin Property when its automorphism group does. In Glasner and Weiss [2001] and Akin, Hurley and Kennedy [2003] it is proved that Cantor space satisfies the Rohlin Property. Under some circumstances the group of μ preserving homeomorphisms satisfies a more powerful condition which we call the *Strong Rohlin Property*:

Theorem 0.2. *If μ is a good measure on X with $S(\mu)$ $\mathbb{Q}$-like, then there exists a single, necessarily unique, conjugacy class which is a dense G_δ subset of the group of measure preserving homeomorphisms on X.*

In such a case, we can speak of *the* generic homeomorphism on X. We show that it is conjugate to the product of the identity on X with the translation map of the universal adding machine.

1. THE CLOPEN VALUES INVARIANTS

All of our topological spaces will be nonempty Polish spaces. That is, they are separable and completely metrizable. Furthermore, unless we mention otherwise, our spaces X, Y are assumed to be compact as well. Such a space is a *Cantor space* when it is zero-dimensional and perfect. One kind of example is a *Cantor set*, a closed, perfect, nowhere dense subset of the unit interval $I = [0, 1]$ in $\mathbb{R}$. If F is a discrete finite space containing at least two points and $\mathbb{Z}_+ = \{0, 1, 2, ...\}$, then the product space $F^{\mathbb{Z}_+}$ is a Cantor space as well.

An *order* $\leq$ on a space X is a total order, i.e. any two points are comparable, which is closed as a subset of $X \times X$, or, equivalently, whose order topology agrees with the original compact topology on X. An *ordered (Cantor) space* is a pair $(X, \leq)$ where X is a (Cantor) space and $\leq$ is an order on X. Thus, $(X, \leq)$ is a compact linearly ordered topological space (acronym: LOTS). Such a space has a maximum element and a minimum element which we will denote by M and m, respectively. We will adopt the usual interval notation so that, for example, $[x, y) = \{z \in X : x \leq z < y\}$.

A point x is called a *left endpoint* if the interval $[m, x]$ is clopen and a *right endpoint* if $[x, M]$ is clopen. Thus, M is a left endpoint and m is a right endpoint. A point is both a left and right endpoint iff it is an isolated point.

If X is perfect, i.e. it has no isolated points, then the endpoints other than m, M occur in pairs. If $[m, x_-]$ is clopen with $x_- \neq M$, then

$$(1.1) \qquad X \setminus [m, x_-] \quad = \quad [x_+, M] \qquad \text{with} \quad x_+ \; = \; min(X \setminus [m, x_-]).$$

Such an *endpoint pair* $\{x_-, x_+\}$ is characterized by the conditions that $x_- < x_+$ but the open interval (x_-, x_+) is empty.

An *order map* $f : (X_1, \leq) \to (X_2, \leq)$ is a continuous map of spaces such that for $x, y \in X_1$

$$(1.2) \qquad \qquad x \leq y \qquad \Longrightarrow \qquad f(x) \leq f(y).$$

If $[a, b]$ is a closed subinterval of X_2, then

$$(1.3) \qquad \begin{aligned} f^{-1}([a, b]) \quad &= \quad [x, y] \qquad \text{with} \\ x \; = \; min f^{-1}([a, b]) \qquad &\text{and} \qquad y \; = \; max f^{-1}([a, b]). \end{aligned}$$

If an order map f is a homeomorphism, then f^{-1} is an order map as well and we call f an *order isomorphism*.

Any two Cantor spaces are homeomorphic and any two ordered Cantor spaces are order isomorphic. We refer to these results as *The Uniqueness of Cantor* (see Akin [1999] Proposition 1.2 and Corollary 2.13 for examples of proofs of these classic results). In particular, any ordered Cantor space is order isomorphic to the classical Cantor set in I. The left and right endpoints of an endpoint pair correspond to the left and right endpoints of one of the open intervals in I complementary to the

Cantor set. Hence, the sets of left and right endpoints are each countable dense subsets of any ordered Cantor space.

A continuous map $f : X_1 \to X_2$ is called *almost open* if for $A \subset X_1$

$$(1.4) \qquad Int A \neq \emptyset \qquad \Longrightarrow \qquad Int f(A) \neq \emptyset.$$

The map is called *almost one-to-one* if Inj_f is dense in X_1 where

$$(1.5) \qquad Inj_f \quad =_{def} \quad \{x \in X_1 : f^{-1}(f(x)) = \{x\}\}.$$

Lemma 1.1. *Let $f : (X_1, \leq) \to (X_2, \leq)$ be a surjective order preserving map with closed point images, i.e. $f^{-1}(y)$ is closed in X_1 for all $y \in X_2$.*

(a) *f is continuous and so is an order map.*
(b) *$f(m_1) = m_2$ and $f(M_1) = M_2$.*
(c) *If y_-, y_+ is an endpoint pair in X_2, then there exists an endpoint pair x_-, x_+ in X_1 such that*

$$(1.6) \qquad f(x_\pm) \quad = y_\pm.$$

(d) *If x_-, x_+ is an endpoint pair in X_1, then either $f(x_-) = f(x_+)$ or $f(x_-)$, $f(x_+)$ is an endpoint pair in X_2.*
(e) *If X_1 and X_2 are perfect, then the following conditions are equivalent:*
   (1) *$X_1 \setminus Inj_f$ is countable.*
   (2) *f is almost one-to-one.*
   (3) *f is almost open.*
   (4) *For all $z \in X_2$, $f^{-1}(z)$ is nowhere dense.*
   (5) *For all $z \in X_2$, $f^{-1}(z)$ is finite.*
   (6) *For all $x, y \in X_1$, $f(x) = f(y)$ implies either $x = y$ or x, y is an endpoint pair.*

Proof. (a) Observe first that since f is surjective we can sharpen (1.3):

$$(1.7) \qquad \begin{aligned} f^{-1}([a,b]) \quad &= \quad [x,y] \qquad \text{with} \\ x \ = \ min f^{-1}(a) \quad &\text{and} \quad y \ = \ max f^{-1}(b), \end{aligned}$$

and similarly,

$$(1.8) \qquad \begin{aligned} f^{-1}((a,b)) \quad &= \quad (\tilde{x}, \tilde{y}) \qquad \text{with} \\ \tilde{x} \ = \ max f^{-1}(a) \quad &\text{and} \quad \tilde{y} \ = \ min f^{-1}(b). \end{aligned}$$

It follows that f is continuous and so is an order map.

(b) If $A \subset X_1$ is closed, then

$$(1.9) \qquad f(max A) \quad = \quad max f(A) \qquad \text{and} \qquad f(min A) \quad = \quad min f(A).$$

Since f is surjective, (b) follows.

(c) From (1.8) this follows with

$$(1.10) \qquad x_- \ = \ max f^{-1}(y_-) \qquad \text{and} \qquad x_+ \ = \ min f^{-1}(y_+).$$

(d) If the interval $(f(x), f(y))$ is nonempty, then there exists $z \in X_1$ with $f(x) < f(z) < f(y)$. Because f is an order map, $x < z < y$ and so x, y is not an endpoint pair.

(e) (4)$\Rightarrow$ (6): Assume $x < y$ with $z = f(x) = f(y)$ but x, y is not an endpoint pair. Since f is an order map, $f^{-1}(z)$ contains the nonempty open interval (x, y). Since X_2 is perfect, $Int\{z\} = \emptyset$. Hence, f is not almost open.

$(6) \Rightarrow (5)$: Obvious.

$(5) \Rightarrow (4)$: Obvious since X_1 is perfect.

$(6) \Rightarrow (1)$: $X_1 \setminus Inj_f$ is a subset of the countable set of endpoints.

$(1) \Rightarrow (2)$: Since X_1 is perfect and compact, the Baire Category Theorem implies that the complement of a countable set is dense.

$(2) \Rightarrow (3)$: Any continuous, almost one-to-one map is almost open. In fact, from compactness it is easy to check that if $U \subset X_1$ is open, then

$$(1.11) \qquad f(U \cap Inj_f) \quad \subset \quad Int f(U).$$

$(3) \Rightarrow (4)$: Because X_2 is perfect, (1.4) implies that $f^{-1}(z)$ is nowhere dense. $\quad\square$

Remark. The unit interval I has no endpoint pairs. Hence, for $f : (X, \leq) \to (I, \leq)$ an almost one-to-one surjection with X perfect, if $x < y$ in X, then

$$(1.12) \qquad f(x) \quad = \quad f(y) \qquad \Longleftrightarrow \qquad \{x, y\} \text{ is an endpoint pair in } X.$$

Following Akin [1999] we study measures on Cantor spaces. A measure μ on a space X is a Borel probability measure, i.e. $\mu(X) = 1$. The measure is *full* if nonempty open sets have positive measure and *nonatomic* if countable sets have zero measure. Let $\mathcal{M}_X$ denote the set of full, nonatomic measures on X. For example, the Lebesgue measure λ is an element of $\mathcal{M}_I$. Notice that the existence of a full, nonatomic measure implies that the space X is perfect.

When we speak of a metric d on any metrizable topological space Z we mean a metric whose associated topology is original topology on Z. If $A \subset Z$ we define the *diameter* of A by

$$(1.13) \qquad d(A) \quad = \quad sup\{d(x, y) : x, y \in A\},$$

with $d(\emptyset) = 0$ by convention.

Lemma 1.2. *Let d be a metric on a space X and let μ be a nonatomic measure on X. For every $\epsilon > 0$ there exists a $\delta > 0$ such that for every Borel set $A \subset X$*

$$(1.14) \qquad d(A) \quad < \quad \delta \qquad \Longrightarrow \qquad \mu(A) \quad < \quad \epsilon.$$

Proof. Since μ is nonatomic, every point $x \in X$ is contained in some open set U_x with $\mu(U_x) < \epsilon$. By compactness, the open cover $\{U_x : x \in X\}$ has a positive Lebesgue number δ. That is, $d(A) < \delta$ implies $A \subset U_x$ for some x. $\quad\square$

If $(X, \leq)$ is an ordered space and $\mu \in \mathcal{M}_X$, then the *cumulative distribution function*, hereafter the CDF, of μ is the order map $F_\mu : (X, \leq) \to (I, \leq)$ defined by

$$(1.15) \qquad F_\mu(x) \quad =_{def} \quad \mu([m, x]).$$

Since μ is nonatomic we have for $x < y$ in X

$$(1.16) \qquad \mu((x, y)) \quad = \quad \mu([x, y]) \quad = \quad F_\mu(y) - F_\mu(x).$$

In particular, if x_-, x_+ is an endpoint pair in X, then

$$(1.17) \qquad F_\mu(x_-) \quad = \quad F_\mu(x_+).$$

Proposition 1.3. *If $(X, \leq)$ is a perfect, ordered space, then the association $\mu \mapsto F_\mu$ defines a bijection from the set $\mathcal{M}_X$ of full, nonatomic measures on X to the set of almost one-to-one surjections of $(X, \leq)$ onto $(I, \leq)$.*

Proof. When the measure μ is nonatomic, it is easy to check that the CDF is continuous and surjective. Because the measure is full, the CDF is almost open and so, by Lemma 1.1(e) it is almost one-to-one. Begining with a continuous, almost one-to-one surjection one defines the measure on intervals via (1.16) and, then applies the usual Lebesgue-Stieltjes construction. See Akin [1999] Propositions 2.5 and 2.6. $\qquad\square$

If $f : X_1 \to X_2$ is a continuous map and μ is a measure on X_1, then the induced measure $f_*\mu$ on X_2 is defined by

$$(1.18) \qquad\qquad f_*\mu(B) \quad =_{def} \quad \mu(f^{-1}(B))$$

for every Borel subset B of X_2.

Assume $\mu \in \mathcal{M}_{X_1}$. If f is surjective, then $f_*\mu$ is full and if $f^{-1}(z)$ is countable for every $z \in X_2$, then $f_*\mu$ is nonatomic. If both these conditions on f hold, then $f_*\mu \in \mathcal{M}_{X_2}$. In particular, any homeomorphism induces a bijection from $\mathcal{M}_{X_1}$ to $\mathcal{M}_{X_2}$. If $(X, \leq)$ is a perfect ordered space and $\mu \in \mathcal{M}_X$, then by Akin [1999] Corollary 2.8, the CDF F_μ maps μ to Lebesgue measure λ on I. That is,

$$(1.19) \qquad\qquad F_{\mu*}\mu = \quad \lambda.$$

Following Akin [1999] we define invariants by using the clopen subsets. For a space X and $\mu \in \mathcal{M}_X$ define the *clopen values set for μ*

$$(1.20) \qquad\qquad S(\mu) \quad =_{def} \quad \{\mu(U) : U \text{ is clopen in } X\}.$$

If $(X, \leq)$ is an ordered space, then the *special clopen values set for μ* is

$$(1.21) \qquad\qquad \tilde{S}(\mu) \quad =_{def} \quad \{F_\mu(x) : x \text{ is an endpoint of } X\}.$$

For any $A \subset \mathbb{R}$ let $G[A]$ denote the additive group generated by A, so that $G[A]$ is the set of all finite sums of differences of elements of A (including $0 =$ the empty sum so that $G[\emptyset] = \{0\}$).

Proposition 1.4.   (a) *Let X be a space and $\mu \in \mathcal{M}_X$. $S(\mu)$ is a countable subset of I containing $0, 1$. If X is a Cantor space, then $S(\mu)$ is dense in I.*

    (b) *Let $(X, \leq)$ be an ordered space and $\mu \in \mathcal{M}_X$. $\tilde{S}(\mu)$ is a countable subset of I containing $0, 1$. If X is a Cantor space, then $\tilde{S}(\mu)$ is dense in I. Furthermore,*

$$(1.22) \qquad\qquad \tilde{S}(\mu) \quad \subset \quad S(\mu) \quad \subset \quad G[\tilde{S}(\mu)] \cap I.$$

Proof. (a) Since X has a countable base and any clopen subset is a finite union of elements of the base, there are only countably many clopen subsets of X, including $\emptyset$ and X. If X is a Cantor space, then we can choose an order, by Uniqueness of Cantor, and then apply (b) to obtain density of $S(\mu)$.

(b) $F_\mu(m) = 0, F_\mu(M) = 1$ and for any left endpoint x, $F_\mu(x) = \mu(U)$, where $U = [m, x]$ is clopen. Hence $\tilde{S}(\mu) \subset S(\mu)$ and so is countable by (a). Every clopen set is a finite disjoint union of clopen intervals and so by (1.16) $S(\mu) \subset G[\tilde{S}(\mu)]$. If X is a Cantor space, then the set of endpoints is dense in X and so its image $\tilde{S}(\mu)$ under the surjection F_μ is dense in I. $\qquad\square$

Proposition 1.5. *For $\alpha = 1, 2$ let X_α be a space and $\mu_\alpha \in \mathcal{M}_{X_\alpha}$. Let $f : X_1 \to X_2$ be a continuous map. If $f_* \mu_1 = \mu_2$, then f is a surjection and*

$$(1.23) \qquad\qquad S(\mu_2) \quad \subset \quad S(\mu_1)$$

with equality if f is a homeomorphism.

Proof. The preimage of $X_2 \setminus f(X_1)$ is empty and so $X_2 \setminus f(X_1)$ is an open set with μ_2 measure zero. Since μ_2 is full, f is surjective. Since the preimage of a clopen set is clopen, (1.23) follows. If f is a homeomorphism, then we apply (1.23) to f^{-1} to get equality. $\qquad\square$

Thus, $S(\mu)$ is a homeomorphism invariant for the measure μ. It is not a complete invariant in general. On the other hand, $\tilde{S}(\mu)$ is a complete order isomorphism invariant. For completeness we will sketch the proof of this and of some other, related, results from Akin [1999].

Theorem 1.6. *For $\alpha = 1, 2$ let $(X_\alpha, \leq)$ be a space and $\mu_\alpha \in \mathcal{M}_{X_\alpha}$.*
   (a) *For a continuous map $f : X_1 \to X_2$ the following conditions are equivalent:*
      (1) *f is an order map and $f_* \mu_1 = \mu_2$.*
      (2) *$F_{\mu_1} = F_{\mu_2} \circ f$.*
      If these conditions hold, then f is an almost one-to-one surjection and

$$(1.24) \qquad\qquad \tilde{S}(\mu_2) \quad \subset \quad \tilde{S}(\mu_1)$$

   with equality if f is a homeomorphism.
   (b) *If $\tilde{S}(\mu_2) \subset \tilde{S}(\mu_1)$, then there is a unique continuous map $f : X_1 \to X_2$ such that $F_{\mu_1} = F_{\mu_2} \circ f$. If $\tilde{S}(\mu_2) = \tilde{S}(\mu_1)$, then this map is a homeomorphism.*

Proof. The closed subset $F = (F_{\mu_2})^{-1} \circ F_{\mu_1} \subset X_1 \times X_2$ can be regarded as a relation from X_1 to X_2. It is a surjective relation, that is, $F(x) \neq \emptyset$ for all $x \in X_1$ and $F^{-1}(y) \neq \emptyset$ for all $y \in X_2$. Furthermore, on the complement of the countable set

$$(1.25) \qquad\qquad S^* \quad =_{def} \quad (F_{\mu_1})^{-1}(\tilde{S}(\mu_2) \cup \tilde{S}(\mu_1))$$

the relation F restics to a well-defined, injective function. In fact, if $x < y$ in $X_1 \setminus S^*$, then

$$(1.26) \qquad F_{\mu_2}(F(x)) \quad = \quad F_{\mu_1}(x) \quad < \quad F_{\mu_1}(y) \quad = \quad F_{\mu_2}(F(y)),$$

and so $F(x) < F(y)$. Since X_1 and X_2 are perfect, $X_1 \setminus S^*$ is dense in X_1 and its image, which also has a countable complement, is dense in X_2.

(a), (2) $\Rightarrow$ (1): Condition (2) implies that the relation F is equal to f on $X_1 \setminus S^*$. Because it is continuous and is order preserving on the dense set $X_1 \setminus S^*$, f is an order map. Because it has a dense image, f is surjective. Since $X_1 \setminus S^* \subset Inj_f$, f is almost one-to-one. Furthermore, there is at most one continuous map which satisfies (2). This proves uniqueness in (b). Furthermore, condition (2) implies that μ_2 and $f_* \mu_1$ agree on all intervals of the form $[m, x]$ and so by (1.16) on all intervals. Since the intervals generate the Borel sets, $f_* \mu_1 = \mu_2$. Finally, Lemma 1.1(c) and (2) imply (1.24).

(1) $\Rightarrow$ (2): This is an easy exercise. See, e.g., Akin [1999] Lemma 2.7. As before when f is a homeomorphism we obtain equality by applying the result to f^{-1}.

(b) Since uniqueness has already been established in the proof of part (a), the remainder is the Lifting Lemma 2.9 of Akin [1999]. The key step is the extension of $f = F$ on $X_1 \setminus S^*$. Each endpoint pair of X_1 is in S^* and is associated by F either

to single point or an endpoint pair in X_2. By assumption (1.24) each endpoint pair in X_2 comes from an endpoint pair in X_1. The extension maps the left endpoint in X_1 to the left endpoint in X_2 and the right to the right. $\square$

Let $\mathcal{D}$ denote the uncountable set of countable, dense subsets of the unit interval I which contain $0, 1$. Let $\mathcal{G}$ denote the automorphism group of $(I, \leq)$, that is, the group of order isomorphisms from $(I, \leq)$ to itself. So $\phi \in \mathcal{G}$ when ϕ is an increasing real-valued funtion on I which fixes 0 and 1. By Lemma 1.1(d), if ϕ is an almost one-to-one, surjective order map on $(I, \leq)$, then $\phi \in \mathcal{G}$.

Theorem 1.7. *Let $(X, \leq)$ be an ordered Cantor space.*

(a) *The group $\mathcal{G}$ acts on the set $\mathcal{M}_X$. For $(\phi, \mu_1) \in \mathcal{G} \times \mathcal{M}_X$ and $\mu_2 \in \mathcal{M}_X$*

$$(1.27) \qquad \mu_2 \quad = \quad \phi\mu_1 \qquad \Longleftrightarrow \qquad F_{\mu_2} = \phi \circ F_{\mu_1}.$$

The action is transitive and free, i.e. for $\mu_1 \in \mathcal{M}_X$ the map $\phi \mapsto \phi\mu_1$ is a bijection from $\mathcal{G}$ to $\mathcal{M}_X$.

(b) *The group $\mathcal{G}$ acts on the set $\mathcal{D}$. For $(\phi, D_1) \in \mathcal{G} \times \mathcal{D}$*

$$(1.28) \qquad D_2 \quad = \quad \phi D_1 \qquad \Longleftrightarrow \qquad D_2 = \phi(D_1).$$

The action is transitive, i.e. for $D_1 \in \mathcal{D}$ the map $\phi \mapsto \phi D_1$ is a surjection from $\mathcal{G}$ to $\mathcal{D}$.

(c) *The map $\tilde{S} : \mathcal{M}_X \to \mathcal{D}$ is $\mathcal{G}$ equivariant, i.e. for $(\phi, \mu) \in \mathcal{G} \times \mathcal{M}_X$*

$$(1.29) \qquad \tilde{S}(\phi\mu) \quad = \quad \phi\tilde{S}(\mu).$$

$\tilde{S}$ induces a bijection between the order isomorphism classes in $\mathcal{M}_X$ and the elements of $\mathcal{D}$.

Proof. This result is Theorem 3.2 together with Corollary 3.3 of Akin [1999]. We sketch the proof of some of the results. The action in (a) is well defined by Proposition 1.3. Transitivity of the action in (b) follows from the classical result that any countable dense subset of I which contians 0 and 1 is order isomorphic to the set of rationals in I. The map $\tilde{S}$ in (c) is obviously equivariant. Since Theorem 1.6 implies that $\tilde{S}(\mu)$ provides a complete order isomorphism invariant for μ, it follows that each $(\tilde{S})^{-1}(D)$ is an order isomorphism class. $\square$

For a space X let $H(X)$ denote the automorphism group of the space, i.e. the group of homeomorphisms on X. Equipped with the topology of uniform convergence, $H(X)$ is a Polish group, that is, a completely metrizable, separable topological group. If μ is a measure on X, then

$$(1.30) \qquad H_\mu(X) \quad =_{def} \quad \{f \in H(X) : f_*\mu \ = \ \mu\}$$

is a closed subgroup and so is itself a Polish group.

In Akin [1999] we described a family of bad measures on a Cantor space X. This was an uncountable subset $\mathcal{M}^*$ of $\mathcal{M}_X$ such that for $\mu_1, \mu_2 \in \mathcal{M}^*$ and $f \in H(X)$

$$(1.31) \qquad f_*\mu_1 \ = \ \mu_2 \qquad \Longleftrightarrow \qquad \mu_1 \ = \ \mu_2 \quad \text{and} \quad f \ = \ 1_X.$$

That is, no two distinct measures in $\mathcal{M}^*$ are homeomorphic and for each $\mu \in \mathcal{M}^*$, the group $H_\mu(X)$ is trivial.

In this paper we consider good measures on a Cantor space X.

2. Good measures on Cantor space

We begin with a bit of algebra.

Let S be a subset of the unit interval I. We call S *group-like* if

$$(2.1) \qquad\qquad S \;\; = \;\; G \cap I$$

for G an additive subgroup of $\mathbb{R}$.

If (2.1) holds with G a $\mathbb{Q}$ vector subspace of $\mathbb{R}$ where $\mathbb{Q}$ is the field of rationals, then we call S $\mathbb{Q}$-*like*. If (2.1) holds with G a subfield of $\mathbb{R}$ we will call S *field-like*.

For subsets A, B of $\mathbb{R}$ we will write $AB = \{ab : a \in A \text{ and } b \in B\}$ and $A + B = \{a + b : a \in A \text{ and } b \in B\}$. In particular, if $\mathbb{Z}$ is the group of integers, then

$$(2.2) \qquad\qquad S + \mathbb{Z} \;\; = \;\; \{\alpha + n : \alpha \in S \text{ and } n \in \mathbb{Z}\}.$$

For G an additive subgroup of $\mathbb{R}$ we will call a positive real number α a *divisor* of G if

$$(2.3) \qquad\qquad \alpha G \;\; = \;\; G.$$

That is, $\alpha\beta \in G$ iff $\beta \in G$. Let $Div(G)$ denote the set of divsors of G. Clearly, $Div(G)$ is a multiplicative subgroup of the multiplicative group $\mathbb{R}^*$ of positive reals. Furthermore,

$$(2.4) \qquad\qquad \mathbb{Z} \;\; \subset \;\; G \;\; \implies \;\; Div(G) \;\; \subset \;\; G.$$

Lemma 2.1. *Let S be a subset of I with $0, 1 \in S$. Let $G[S]$ be the additive subgroup of $\mathbb{R}$ generated by S.*

  (a) $\quad S \subset S + \mathbb{Z} \subset G[S] \qquad and \qquad S = (S + \mathbb{Z}) \cap I.$
  (b) *The following conditions on S are equivalent:*
    (1) *S is group-like.*
    (2) *$S + \mathbb{Z} = G[S]$.*
    (3) *$S + \mathbb{Z}$ is an additive subgroup of $\mathbb{R}$.*
    (4) *$\alpha, \beta \in S$ and $\alpha \le \beta$ imply that $\beta - \alpha \in S$.*
    If S is group-like and G is an additive subgroup of $\mathbb{R}$, then

$$(2.5) \qquad\qquad S \;\; = \;\; G \cap I \qquad \Longleftrightarrow \qquad S + \mathbb{Z} \;\; = \;\; G.$$

  (c) *When S is group-like we let $Div(S)$ denote $Div(S + \mathbb{Z})$. For $\alpha \in I \setminus \{0\}$*

$$(2.6) \qquad\qquad \alpha \in Div(S) \qquad \Longleftrightarrow \qquad \alpha S \;\; = \;\; S \cap [0, \alpha].$$

  (d) *The following conditions on S are equivalent:*
    (1) *S is $\mathbb{Q}$-like.*
    (2) *$S + \mathbb{Z}$ is a $\mathbb{Q}$ vector subspace of $\mathbb{R}$.*
    (3) *S is group-like and every positive integer is a divisor of $S + \mathbb{Z}$.*
    When S is $\mathbb{Q}$-like, then $\mathbb{Q} \cap I \subset S$.
  (e) *The following conditions on S are equivalent:*
    (1) *S is field-like.*
    (2) *$S + \mathbb{Z}$ is a subfield of $\mathbb{R}$.*
    (3) *S is $\mathbb{Q}$-like and every positive element of S is a divisor of $S + \mathbb{Z}$.*
    (4) *S is $\mathbb{Q}$-like and for every $\alpha \in S$*

$$(2.7) \qquad\qquad \alpha S \;\; = \;\; S \cap [0, \alpha].$$

  (f) *S is field-like iff all of the following five properties hold:*
    (i) $\quad \mathbb{Q} \cap I \;\; \subset \;\; S.$
    (ii) $\quad \alpha \in S \;\; \implies \;\; 1 - \alpha \in S.$

(iii) $\alpha, \beta \in S \implies \alpha\beta, (\alpha + \beta)/2 \in S.$
(iv) $\alpha, \beta \in S$ and $\alpha + \beta \neq 0 \implies \alpha/(\alpha + \beta) \in S.$
(v) $\alpha \in S$ and $\alpha < 1/2 \implies 2\alpha \in S.$

Proof. (a) Since $0, 1 \in S$ the inclusions are clear. On the other hand, if $\alpha + n \in I$ with $\alpha \in S$ and $n \in \mathbb{Z} \setminus \{0\}$, then either $n = 1$ and $\alpha = 0$ or $n = -1$ and $\alpha = 1$. Either way, $\alpha + n \in \{0, 1\}$. Hence the equation follows.

(b) (1)$\Rightarrow$(4): Obvious.

(4)$\Rightarrow$(3): If $\alpha \leq \beta$ in S, then $\beta - \alpha \in S$ by (4) and $\alpha - \beta + 1 = 1 - (\beta - \alpha) \in S$ by (4) applied twice. It easily follows that $S + \mathbb{Z}$ is closed under subtraction and so is an additive subgroup of $\mathbb{R}$.

(3)$\Rightarrow$(2): Obvious from the inclusions in (a).

(2)$\Rightarrow$(1): By the equality in (a), (2.1) holds with $G = G[S]$.

In general, if $S = G \cap I$ for some subgroup G of $\mathbb{R}$, then by (a) $\mathbb{Z} + (G \cap I) = S + \mathbb{Z} \subset G$. On the other hand, if $1 \in G$ and G is a subgroup of $\mathbb{R}$, then any element of G is of the form $n + \alpha$ with $\alpha \in G \cap I$. That is, $G \subset \mathbb{Z} + (G \cap I)$.

(c) Multiplying the equation in (a) by any positive real α we get

$$(2.8) \qquad\qquad \alpha S \quad = \quad (\alpha(S + \mathbb{Z})) \cap [0, \alpha]$$

If, in addition, $\alpha \in I$, then

$$(2.9) \qquad S \cap [0, \alpha] \quad = (S + \mathbb{Z}) \cap I \cap [0, \alpha] \quad = \quad (S + \mathbb{Z}) \cap [0, \alpha].$$

Thus, for $\alpha \in Div(S)$, $\alpha(S + \mathbb{Z}) = S + \mathbb{Z}$ implies $\alpha S = S \cap [0, \alpha]$.

Conversely, if $\alpha S = S \cap [0, \alpha]$, then (a) and (2.9) imply that $(S + \mathbb{Z}) \cap I = S = (\alpha^{-1} S \cap I) = (\alpha^{-1}(S + \mathbb{Z}) \cap I)$. Since $\alpha^{-1}(S + \mathbb{Z})$ is an additive subgroup of $\mathbb{R}$ it follows from (2.5) that $S + \mathbb{Z} = \alpha^{-1}(S + \mathbb{Z})$ and so $\alpha^{-1} \in Div(S)$ which implies $\alpha \in Div(S)$.

(d) (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3): Obvious from (b).

(3) $\Rightarrow$ (2): Since S is group-like $S + \mathbb{Z}$ is an additive group. Since every positive integer is a divisor, it is a rational vector space.

Since $1 \in S$, (2) implies $\mathbb{Q} \subset S + \mathbb{Z}$. Intersect with I and apply the equation in (a) to complete the proof of (d).

(e) (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3): Obvious from (b).

(3) $\Rightarrow$ (4) : Obvious from (c).

(4)$\Rightarrow$ (2): $G = S + \mathbb{Z}$ is a $\mathbb{Q}$ vector space and from (2.7) we obtain $\beta G \subset G$ for any $\beta \in S$ from which it easily follows that G is closed under multiplication. Given $g_1, g_2 \in G$ with $g_2 > 0$ write the ratio g_1/g_2 as $\beta + n$ with $n \in \mathbb{Z}$ and $1 \geq \beta > 0$. We have $\beta = g_3/g_2$ with $g_3 = g_1 - ng_2$ a positive element of G. Choose a positive integer N greater than both g_1 and g_3. Replacing g_k by g_k/N for $k = 1, 3$ we remain in the rational vector space G. So we can assume that $g_1, g_3 \in G \cap I = S$. Apply (2.7) with $\alpha = g_3$ to get that $\beta \in S$. Hence the ratio $g_1/g_2 = \beta + n \in G$. It follows that G is a field.

(f) If S is field-like, then conditions (i)-(v) obviously hold. Now assume conditions (i)-(v). We will prove that $G = S + \mathbb{Z}$ is a field.

Observe first that if $\alpha, \beta \in S$ and $\alpha + \beta \leq 1$, then (iii) and (v) imply that $\alpha + \beta \in S$. If $\alpha \leq \beta$, then $(1 - \beta) + \alpha \leq 1$ and so by (ii), (iii) and (v) $(1 - \beta) + \alpha \in S$. By (ii) $\beta - \alpha = 1 - ((1 - \beta) + \alpha) \in S$. From part (b) it follows that S is group-like and G is a group. By (iii) S is closed under multiplication from which it easily follows that $G = S + \mathbb{Z}$ is closed under multiplication. From (i) it follows that $\mathbb{Q} \subset S + \mathbb{Z}$ and so G is a rational vector space and S is $\mathbb{Q}$-like.

By (e) it suffices to prove (2.7) for a positive α in S. Since S is closed under multiplication by (iii), $\alpha S \subset S \cap [0, \alpha]$. On the other hand, if $\beta \in S$ and $\beta \leq \alpha$, then

$$(2.10) \qquad \beta/\alpha \quad = \quad \beta/(\beta + (\alpha - \beta))$$

and so $\beta/\alpha \in S$ by (iv). Thus, (2.7) holds and so G is a field by (e). $\qquad \square$

We will call a positive integer n a *reciprocal* for S if $1/n \in S$. Let $Rec(S)$ denote the set of reciprocals in S. Clearly,

$$(2.11) \qquad Div(S) \cap \mathbb{Z}_+ \quad \subset \quad Rec(S).$$

Lemma 2.2. *Let S be a group-like subset of I with $0, 1 \in S$.*

If $m/n \in S$ with m, n relatively prime positive integers, then $n \in Rec(S)$. If $n \in Rec(S)$ and $m|n$, then $m \in Rec(S)$. If $m, n \in Rec(S)$, then the least common multiple, $lcm(m, n)$, is in $Rec(S)$.

In general,

$$(2.12) \qquad \mathbb{Q} \cap S \quad = \quad \{k/n : k = 0, 1, ..., n \ \ and \ \ n \in Rec(S)\}.$$

If $Rec(S)$ is infinite, then $\mathbb{Q} \cap S$ is dense in I. If $Rec(S)$ is finite, then $\mathbb{Q} \cap S$ is finite.

Proof. Observe that for $x, y \in \mathbb{Z}$ and positive integers m, n:

$$(2.13) \qquad \begin{aligned} x(m/n) \ + \ y(1) \quad &= \quad (xm + yn)/n, \\ x(1/n) \ + \ y(1/m) \quad &= \quad (xm + yn)/mn. \end{aligned}$$

Given such m, n we can choose x, y so that $xm + yn = gcd(m, n)$, the greatest common divisor. So if $m/n \in S$ with m, n relatively prime, then $1/n \in S$ because S is group-like. Equation (2.12) clearly follows and implies that any divisor of an element of $Rec(S)$ is in $Rec(S)$. Similarly, if the reciprocals of m and n are in S, then the second equation in (2.13) implies that $1/lcm(m, n) = gcd(m, n)/mn \in S$. Finally, from (2.12) it is clear that if $Rec(S)$ contains arbitrarily large integers, then $\mathbb{Q} \cap S$ is dense in I and that otherwise the intersection is finite. $\qquad \square$

Remark. Notice that if $Rec(S)$ is infinite and S is group-like with $0, 1 \in S$, then $Rec(S)$ is directed with respect to the partial order of divisibility, i.e. $m, n \in Rec(S)$ implies there exists $k \in Rec(S)$ such that $m|k$ and $n|k$, namely $k = lcm(m, n)$.

Definition 2.3. Let X be a Cantor space. We say that a measure μ on X satisfies the *Subset Condition* if whenever U and V are clopen subsets of X with $\mu(U) \leq \mu(V)$ there exists a clopen subset U_1 such that

$$(2.14) \qquad U_1 \subset V \qquad and \qquad \mu(U_1) \ = \ \mu(U).$$

A measure μ on a Cantor space X is called a *good measure* when $\mu \in \mathcal{M}_X$, i.e. it is full and nonatomic, and, in addition, μ satisfies the Subset Condition.

For a clopen subset V of X we relativize the clopen values set, defining

$$(2.15) \qquad S(\mu, V) \quad =_{def} \quad \{\mu(U) : U \text{ is clopen in } X \quad and \quad U \subset V\}.$$

Clearly, we always have

$$(2.16) \qquad S(\mu, V) \quad \subset \quad S(\mu) \cap [0, \mu(V)].$$

The Subset Condition says exactly that equality holds in (2.16). That is, μ satisfies the Subset Condition iff for every clopen subset V of X

$$(2.17) \qquad S(\mu, V) \quad = \quad S(\mu) \cap [0, \mu(V)].$$

For a measure $\mu \in \mathcal{M}_X$ and V a nonempty clopen subset of X we have $\mu(V) > 0$ and so we can define the measure $\mu_V \in \mathcal{M}_V$ by

$$(2.18) \qquad \mu_V(A) \quad =_{def} \quad \mu(A)/\mu(V)$$

for every Borel subset $A \subset V$. We regard μ_V as a measure on X by

$$(2.19) \qquad \mu_V(B) \quad =_{def} \quad \mu_V(B \cap V) \quad = \quad \mu(B \cap V)/\mu(V)$$

for every Borel subset $B \subset X$. Regarded as a measure on X, μ_V is nonatomic but not full if V is a proper subset of X.

Clearly, we have

$$(2.20) \qquad S(\mu_V) \quad = \quad (1/\mu(V))S(\mu, V).$$

It follows from (2.17) that μ is a good measure iff $\mu \in \mathcal{M}_X$ and for every nonempty clopen subset V of X

$$(2.21) \qquad S(\mu_V) \quad = \quad [(1/\mu(V))S(\mu)] \cap I.$$

Proposition 2.4. *If μ is a good measure on a Cantor space X, then the clopen values set $S(\mu)$ is group-like. If, in addition, V is a nonempty clopen subset of X, then μ_V is a good measure on the Cantor space V and $S(\mu_V)$ is group-like.*

Proof. Assume $\alpha = \mu(U) \leq \mu(V) = \beta$ for clopen subsets U, V. Apply the Subset Condition to get a clopen U_1 which satisfies (2.14). Since $V \setminus U_1$ is clopen,

$$(2.22) \qquad \beta - \alpha \quad = \quad \mu(V \setminus U_1) \quad \in \quad S(\mu).$$

From Lemma 2.1(b) it follows that $S(\mu)$ is group-like.

It is clear that μ_V is nonatomic and full on V. The Subset Condition for μ easily implies the Subset Condition for μ_V on V. Hence, μ_V is good and so its clopen values set is group-like. $\qquad\square$

Definition 2.5. Let $(X, \leq)$ be an ordered Cantor space and $\mu \in \mathcal{M}_X$. We say that μ is *adapted* to $(X, \leq)$, or that the order $\leq$ is *adapted* to μ, when

$$(2.23) \qquad \tilde{S}(\mu) \quad = \quad S(\mu).$$

Theorem 2.6. *Let $(X, \leq)$ be an ordered Cantor space and $\mu \in \mathcal{M}_X$.*

- (a) *The measure μ is adapted to $(X, \leq)$ iff the special clopen values set $\tilde{S}(\mu)$ is a group-like subset of I.*
- (b) *If μ is adapted to $(X, \leq)$, then μ satisfies the Subset Condition and so is a good measure on X.*
- (c) *If μ is adapted to $(X, \leq)$ and V is a nonempty clopen subset of X, then μ_V is adapted to the ordered Cantor space $(V, \leq)$, with the order induced from $(X, \leq)$.*

Proof. (a) If $\tilde{S}(\mu)$ is group-like, then by Lemma 2.1(a),(b) $\tilde{S}(\mu) = G[\tilde{S}(\mu)] \cap I$ and so by (1.22) $\tilde{S}(\mu) = S(\mu)$. On the other hand, suppose that $0 < \alpha \leq \beta \in \tilde{S}(\mu)$. There exist left endpoints x, y in X such that $\alpha = \mu([0, x])$ and $\beta = \mu([0, y])$. Since μ is full $x \leq y$ in X. Since $(x, y]$ is clopen we have $\beta - \alpha = \mu((x, y]) \in S(\mu)$. So if $\tilde{S}(\mu) = S(\mu)$, then $\beta - \alpha \in \tilde{S}(\mu)$ and so $\tilde{S}(\mu)$ is group-like by Lemma 2.1(b).

(b) If V is a nonempty clopen subset of X, then it can be written as a finite, disjoint union of clopen intervals:

(2.24)
$$V = \bigcup_{i=1}^{k} V_i \quad \text{with} \quad V_i = [x_i^+, y_i^-]$$
$$\text{such that} \quad x_i^+ < y_i^- < x_{i+1}^+$$

and x_i^+ is a right endpoint and y_i^- is a left endpoint for all i. Define $y_0^+ = m$ and let

(2.25)
$$W_i \quad =_{def} \quad [y_{i-1}^+, x_i^-] \quad \text{for } i = 1, ..., k,$$

where x_i^-, x_i^+ and y_i^+, y_i^- are endpoint pairs. If $x_1^+ = m$, then $W_1 =_{def} \emptyset$.

Now let U be a nonempty clopen subset of X such that $0 < \mu(U) \leq \mu(V)$. Define $r \in \{1, ..., k\}$ so that

(2.26)
$$\sum_{j=1}^{r-1} \mu(V_j) \quad < \quad \mu(U) \quad \leq \quad \sum_{j=1}^{r} \mu(V_j)$$

and let

(2.27)
$$\tilde{\alpha} \quad = \quad \sum_{j=1}^{r} \mu(W_j) + \mu(U),$$

so that

(2.28)
$$\mu([m, x_r^-]) \quad < \quad \tilde{\alpha} \quad \leq \quad \mu([m, y_r^-]).$$

By (a), $\tilde{S}(\mu) = S(\mu)$ implies that $\tilde{S}(\mu)$ is group-like and so $\tilde{\alpha} \in S(\mu) = \tilde{S}(\mu)$. That is, $\tilde{\alpha} = \mu([m, z])$, where z is a left endpoint of X. Clearly, $x_r^- < z \leq y_r^-$ and so $x_r^+ \leq z$. Hence,

(2.29)
$$U_1 \quad =_{def} \quad [m, z] \cap V \quad = \quad [0, z] \setminus \bigcup_{j=1}^{r} W_j$$

satisfies $\mu(U_1) = \mu(U)$.

Thus, μ satisfies the Subset Condition.

(c) Let α be a positive element of $S(\mu_V)$ so that $\mu(V)\alpha = \mu(U)$ for U some clopen subset of V. We can apply the construction of part (b) to get $U_1 = [0, z] \cap V$ with $\mu(U) = \mu(U_1)$. Clearly, U_1 is a clopen subset of V such that $x \in V$ and $x \leq z$ imply $x \in U_1$. So by definition of $\tilde{S}$, $\alpha = \mu_V(U_1)$ is an element of $\tilde{S}(\mu_V)$. $\square$

The main result of this section is a converse construction using partitions. A *partition* $\mathcal{A}$ of a Cantor space X is a pairwise disjoint cover of X by nonempty clopen subsets. By compactness a partition is finite. Giving $\mathcal{A}$ the discrete topology we define the *carrier map* as the continuous, locally constant surjection $\pi^{\mathcal{A}} : X \to \mathcal{A}$ given by

(2.30)
$$x \in \pi^{\mathcal{A}}(x) \quad \text{for } x \in X.$$

If d is a metric on X and μ is a measure on X we define the *mesh* of $\mathcal{A}$ and the μ *mesh* of $\mathcal{A}$ by

(2.31)
$$mesh(\mathcal{A}) \quad =_{def} \quad max \ \{d(A) : A \in \mathcal{A}\},$$
$$\mu mesh(\mathcal{A}) \quad =_{def} \quad max \ \{\mu(A) : A \in \mathcal{A}\}.$$

A partition $\mathcal{B}$ *refines* $\mathcal{A}$, written $\mathcal{B} \to \mathcal{A}$ if each element of $\mathcal{B}$ is contained in a, necessarily unique, member of $\mathcal{A}$. We define the surjection $\pi_{\mathcal{B}}^{\mathcal{A}} : \mathcal{B} \to \mathcal{A}$ by

$$(2.32) \qquad\qquad B \subset \pi_{\mathcal{B}}^{\mathcal{A}}(B) \qquad \text{for } B \in \mathcal{A}.$$

Clearly,

$$(2.33) \qquad\qquad \pi^{\mathcal{A}} = \pi_{\mathcal{B}}^{\mathcal{A}} \circ \pi^{\mathcal{B}}.$$

Hence, if three partitions satisfy $\mathcal{C} \to \mathcal{B} \to \mathcal{A}$, then

$$(2.34) \qquad\qquad \pi_{\mathcal{C}}^{\mathcal{A}} = \pi_{\mathcal{B}}^{\mathcal{A}} \circ \pi_{\mathcal{C}}^{\mathcal{B}}.$$

An *ordered partition* $(\mathcal{A}, \leq)$ is a partition with a total order on it. If $(\mathcal{A}, \leq)$ is an ordered partition, then we can *order number* $\mathcal{A}$. That is, write

$$(2.35) \qquad \mathcal{A} = \{A_0, ..., A_n\} \qquad \text{with } i < j \quad \Leftrightarrow \quad A_i < A_j.$$

If $(\mathcal{B}, \leq)$ and $(\mathcal{A}, \leq)$ are ordered partitions, then we write $(\mathcal{B}, \leq) \to (\mathcal{A}, \leq)$ if $\mathcal{B} \to \mathcal{A}$ and

$$(2.36) \qquad B_1 \leq B_2 \quad \Longrightarrow \quad \pi_{\mathcal{B}}^{\mathcal{A}}(B_1) \leq \pi_{\mathcal{B}}^{\mathcal{A}}(B_2) \qquad \text{for } B_1, B_2 \in \mathcal{B}.$$

Conversely, if $(\mathcal{A}, \leq)$ is an ordered partition and $\mathcal{B} \to \mathcal{A}$, then by choosing an arbitrary ordering on $(\pi_{\mathcal{B}}^{\mathcal{A}})^{-1}(A)$ for each $A \in \mathcal{A}$ we obtain an ordering on $\mathcal{B}$ so that $(\mathcal{B}, \leq) \to (\mathcal{A}, \leq)$.

Theorem 2.7. *Let μ be a good measure on a Cantor space X and let $x_0 \in X$. There exists an order $\leq$ on X such that the minimum point m of $(X, \leq)$ is x_0 and the measure μ is adapted to $(X, \leq)$, i.e. $\tilde{S}(\mu) = S(\mu)$.*

Proof. Choose a metric d on X. Write $S(\mu) \setminus \{0\} = \{\alpha_0, \alpha_1, \alpha_2, ...\}$ with $\alpha_0 = 1$.

For each $k = 0, 1, 2, ...$ we will construct an ordered partition $(\mathcal{A}_k, \leq)$ and order number it: $\mathcal{A}_k = \{A_{k0}, ..., A_{kn_k}\}$ and we will define $j_k \leq n_k$ so that the following conditions hold:

(i) If $k \geq 1$, then $(\mathcal{A}_k, \leq) \to (\mathcal{A}_{k-1}, \leq)$.
(ii) $mesh(\mathcal{A}_k) \leq 1/k$ and $\mu mesh(\mathcal{A}_k) \leq \alpha_k$.
(iii) $\pi^{\mathcal{A}_k}(x_0)$ is the minimum element A_{k0} of $\mathcal{A}_k$.
(iv) $\alpha_k = \sum \{\mu(A_{kj}) : 0 \leq j \leq j_k\}$.

Begin with $\mathcal{A}_0 = \{X\}$ and $j_0 = 0$. Since $\alpha_0 = 1$, conditions (i)-(iv) hold for $k = 0$. Now proceed inductively assuming that $k > 1$ and $(\mathcal{A}_{k-1}, \leq)$ has been defined.

Because the measure is nonatomic we can partition and order each $A_{(k-1)j}$ separately to obtain an ordered partition $(\mathcal{B}, \leq)$ which refines $(\mathcal{A}_{k-1}, \leq)$, has mesh at most $1/k$ and μ mesh at most α_k. We can choose the ordering on the partition of $A_{(k-1)0}$ so that x_0 is in the minimum element. Thus, as a candidate for $(\mathcal{A}_k, \leq)$, $(\mathcal{B}, \leq)$ satisfies conditions (i)-(iii). Order number $\mathcal{B} = \{B_0, ..., B_n\}$ and define $j*$ so that

$$(2.37) \qquad \sum \{\mu(B_j) : 0 \leq j < j*\} < \alpha_k \leq \sum \{\mu(B_j) : 0 \leq j \leq j*\}.$$

Case i: If $\alpha_k = \sum \{\mu(B_j) : 0 \leq j \leq j*\}$, then let $(\mathcal{A}_k, \leq) = (\mathcal{B}, \leq)$ and $j_k = j*$.

Case ii: If both inequalities are strict, then $\mu mesh(\mathcal{B}) \leq \alpha_k$ implies that $0 < j*$. Because $S(\mu)$ is group-like $\tilde{\alpha} =_{def} \alpha_k - \sum \{\mu(B_j) : 0 \leq j < j*\}$ is a member of $S(\mu)$ and so by the Subset Condition we can choose a clopen subset $U \subset B_{j*}$ with

$\mu(U) = \tilde{\alpha}$. Replace B_{j*} by the two clopen sets $U, B_{j*} \setminus U$. This defines the ordered partition $(\mathcal{A}_k, \leq) \to (\mathcal{B}, \leq)$ when listed in order $\mathcal{A}_k = \{B_0, ..., B_{j*-1}, U, B_{j*} \setminus U, B_{j*+1}, ..., B_n\}$. Thus, $n_k = n + 1$. If we let $j_k = j*$, then condition (iv) holds. Since $0 < j*$ the minimum element is unchanged and so it still contains x_0.

This completes the inductive construction. Now define the order on X by

(2.38) $\qquad x \leq y \qquad \Longleftrightarrow \qquad \pi^{\mathcal{A}_k}(x) \leq \pi^{\mathcal{A}_k}(y) \quad \text{for all } k.$

The relation is clearly reflexive, transitive and closed. Because $(\mathcal{A}_k, \leq)$ refines $(\mathcal{A}_{k-1}, \leq)$ for all k any two elements are comparable. Since the mesh of $\mathcal{A}_k$ tends to zero the relation is anti-symmetric. Thus we have defined a total order on X. From condition (iii) it is clear that x_0 is the minimum element m of X.

Finally, let z_k be the maximum element of the clopen set $A_{k j_k}$. Clearly,

(2.39) $\qquad\qquad\qquad [m, z_k] \quad = \quad \bigcup \{A_{kj} : 0 \leq j \leq j_k\}.$

Since the maximum of a nonempty clopen set is a left endpoint, condition (iv) implies that $\alpha_k \in \tilde{S}(\mu)$. As this is true for all k we have $S(\mu) \subset \tilde{S}(\mu)$. By (1.22) the reverse inclusion is always true. Hence, with this choice of order μ is adapted to $(X, \leq)$. $\qquad\qquad\square$

Corollary 2.8. *Let X be a Cantor space and μ be a full, nonatomic measure on X. The measure μ is good on X, i.e. it satisfies the Subset Condition, iff there exists an ordering $\leq$ on X with respect to which $\tilde{S}(\mu) = S(\mu)$.*

Proof. If such an ordering exists, then μ is good by Theorem 2.6(b). Conversely, if μ is good, then the required ordering exists by Theorem 2.7. $\qquad\qquad\square$

Together with Theorem 2.6(b) this result yields Theorem 0.1 of the Introduction.

It also follows that for good measures the clopen values set is a complete invariant. We will call two measures on a space *homeomorphic* if there is a homeomorphism mapping one to the other. Similarly, we will call two measures on an ordered space *order isomorphic* if there is an order isomorphism mapping one to the other.

Theorem 2.9. *For $k = 1, 2$ let X_k be a Cantor space, μ_k be a measure on X_k and $x_k \in X_k$. Assume that μ_1 is a good measure on X_1.*

(a) *If there exists a homeomorphism $f : X_1 \to X_2$ such that $f_*\mu_1 = \mu_2$, then μ_2 is a good measure on X_2 and $S(\mu_1) = S(\mu_2)$.*

   Conversely, if μ_2 is a good measure on X_2 and $S(\mu_1) = S(\mu_2)$, then there exists a homeomorphism $f : X_1 \to X_2$ such that $f_*\mu_1 = \mu_2$. Furthermore, f can be chosen so that $f(x_1) = x_2$.

   In particular, two good measures are homeomorphic iff they have the same clopen values set.

(b) *If there exists a continuous map $f : X_1 \to X_2$ such that $f_*\mu_1 = \mu_2$, then f is surjective and $S(\mu_1) \supset S(\mu_2)$.*

   Conversely, if μ_2 is a good measure on X_2 and $S(\mu_1) \supset S(\mu_2)$, then there exists a continuous, almost one-to-one surjection $f : X_1 \to X_2$ such that $f_*\mu_1 = \mu_2$. Furthermore, f can be chosen so that $f(x_1) = x_2$.

Proof. (a) It is easy to check that the absence of atoms, fullness and the Subset Condition are homeomorphism invariants. The clopen values set is a homeomorphism invariant by Proposition 1.5.

Conversely, if μ_k is good on X_k, then by Theorem 2.7 we can choose an ordering so that μ_k is adapted to $(X_k, \leq)$ and x_k is the minimum point of X_k. If the clopen values sets agree, then $\tilde{S}(\mu_1) = S(\mu_1) = S(\mu_2) = \tilde{S}(\mu_2)$. By Theorem 1.6 there is a unique order isomorphism $f : (X_1, \leq) \to (X_2, \leq)$ such that $f_*\mu_1 = \mu_2$. Such an order isomorphism maps the minimum point x_1 to the minimum point x_2.

(b) As in (a) the first part follows from Proposition 1.5. For the second we choose orderings as in (a) and apply Theorem 1.6 again. $\qquad\square$

We can rephrase these results using the language of Theorem 1.7. We have let $\mathcal{D}$ denote the uncountable collection of countable dense subsets of I which contain $0, 1$. Let $\mathcal{D}_g, \mathcal{D}_q, \mathcal{D}_f$ denote the subcollections of group-like, $\mathbb{Q}$-like and field-like elements of $\mathcal{D}$. There are uncountably many distinct countable subfields of $\mathbb{R}$ and so each of these collections is uncountable.

Theorem 2.10. *Let $(X, \leq)$ be an ordered Cantor space. The set of measures adapted to $(X, \leq)$ is $\tilde{S}^{-1}(\mathcal{D}_g)$. For measures μ_1, μ_2 adapted to $(X, \leq)$, $\tilde{S}(\mu_1) = \tilde{S}(\mu_2)$ implies that μ_1 and μ_2 are order isomorphic. On the other hand, $\tilde{S}(\mu_1) \neq \tilde{S}(\mu_2)$ implies that μ_1 and μ_2 are not even homeomorphic.*

Proof. The first result is a restatement of Theorem 2.6(a). By Theorem 1.6 the special clopen values sets are order isomorphism invariants. For adapted measures these are the same as the clopen values sets which are homeomorphism invariants by Proposition 1.5. $\qquad\square$

If H is a subgroup of the automorphism group $H(Z)$ of a Polish space Z, then H acts on Z. If $z \in Z$, then $Hz =_{def} \{h(z) : h \in H\}$ is the H *orbit* of z, or, when H is understood, just the *orbit* of z. Two points $z_1, z_2 \in Z$ are called H *equivalent* if they lie in the same H orbit, i.e. there exists $h \in H$ such that $h(z_1) = z_2$. The action is called *transitive* if Z consists of a single orbit, i.e. all points are H equivalent. The action is called *topologically transitive* if there exists a dense orbit. That is, the set

$$(2.40) \qquad Trans(H) \quad =_{def} \quad \{z \in Z : \overline{Hz} = Z\}$$

is nonempty. In that case, it is a dense G_δ subset of the Polish space Z. The action is called *minimal* if every orbit is dense, i.e. $Trans(H) = Z$. Of course, transitive $\Rightarrow$ minimal $\Rightarrow$ topologically transitive.

From Theorem 2.9(a) we see that if μ is a good measure on a Cantor space X, then the group $H_\mu(X)$ of measure preserving homeomorphisms acts transitively on X. We will strengthen this result below.

First, we go back to Subset Condition itself. In it we did not assume that the move from the clopen set U to the clopen $U_1 \subset V$ was effected by a measure automorphism. That is, we did not assume that $U_1 = h(U)$ for some $h \in H_\mu(X)$. In fact, we obtain this apparently stronger condition for free.

Proposition 2.11. *Let μ be a good measure on the Cantor space X.*

(a) *Let U, V be clopen subsets of X with $x \in U$ and $y \in V$. If $\mu(U) \leq \mu(V)$, then there exists $h \in H_\mu(X)$ such that*

$$(2.41) \qquad h(x) \quad = \quad y \qquad and \qquad h(U) \quad \subset \quad V.$$

Furthermore, if $h \in H_\mu(X)$ satisfies $h(U) \subset V$, then $h(U) = V$ iff $\mu(U) = \mu(V)$.

(b) *Let V be a clopen subset of X with $x \in V$. If $\alpha \in S(\mu) \cap (0, \mu(V)]$, then there exists U a clopen subset of X such that*

$$(2.42) \qquad x \in U, \quad U \subset V, \quad and \quad \mu(U) = \alpha.$$

Proof. (a) By the Subset Condition there exists a clopen $U_1 \subset V$ with $\mu(U_1) = \mu(U)$. Let $G = U \setminus (U \cap U_1)$ and $G_1 = U_1 \setminus (U \cap U_1)$. So G and G_1 are disjoint clopens with the same measure.

By Proposition 2.4, the measures μ_{G_1} and μ_G are good and by (2.20) they have the same clopen values set. By Theorem 2.9(a) there is a homeomorphism $f : G \to G_1$ which maps μ_G to μ_{G_1}. Define $h_1 \in H_\mu(X)$ to be f on G, f^{-1} on G_1 and the identity on $X \setminus (G \cup G_1)$. Clearly, $h_1(U) = U_1$. Let $x_1 = h_1(x) \in V$. Now apply Theorem 2.9(a) to the good measure μ_V. There exists a measure preserving automorphism of V which maps x_1 to y. Extend by the identity on $X \setminus V$ to obtain $h_2 \in H_\mu(X)$. Finally, let $h = h_2 \circ h_1$.

If $h \in H_\mu$ and $h(U) \subset V$, then $V \setminus h(U)$ is a clopen subset of X. Since μ is full this difference is empty iff it has measure zero. Thus, $h(U) = V$ iff $\mu(U) = \mu(V)$.

(b) By the subset condition there exists a clopen $U_1 \subset V$ with $\mu(U_1) = \alpha$. Since $\alpha > 0$ there exists $x_1 \in U_1$. By Theorem 2.9(a) there exists a measure preserving automorphism f of V which maps x_1 to x. Let $U = f(U_1)$. $\qquad \square$

Corollary 2.12. *Let μ be a good measure on the Cantor space X. If $\mathcal{A} = \{A_1, ..., A_n\}$ and $\mathcal{B} = \{B_1, ..., B_n\}$ are partitions of X with $\mu(A_i) = \mu(B_i)$ for $i = 1, ..., n$, then there exists $h \in H_\mu(X)$ such that*

$$(2.43) \qquad h(A_i) = B_i \qquad for \quad i = 1, ..., n.$$

Proof. By Proposition 2.11(a) there exists $h_i \in H_\mu(X)$ such that $h_i(A_i) = B_i$ for $i = 1, ..., n$. Let $h = h_i$ on A_i for $i = 1, ..., n$. $\qquad \square$

We now prove a strong homogeneity result.

Theorem 2.13. *Let μ be a good measure on the Cantor space X. If $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_n\}$ are two lists of n distinct points in X, then there exists $h \in H_\mu(X)$ such that $h(x_i) = y_i$ for $i = 1, ..., n$.*

In particular, $H_\mu(X)$ acts transitively on X.

Proof. Since X is perfect we can compare each list to another list of n distinct points and so assume that all $2n$ points are distinct. We can then choose $2n$ pairwise disjoint clopen sets $\{U_1, ..., U_n\}$ and $\{V_1, ..., V_n\}$ so that $x_i \in U_i$ and $y_i \in V_i$ for $i = 1, ..., n$. By applying Proposition 2.11(b) we can shrink them if necessary so that they all have the same measure $\alpha > 0$. Apply Proposition 2.11(a) to obtain a measure preserving homeomorphism $h_i : U_i \to V_i$ which maps x_i to y_i for $i = 1, ..., n$. Define $h \in H_\mu(X)$ to be h_i on U_i, h_i^{-1} on V_i for $i = 1, ..., n$ and the identity on $X \setminus \bigcup_{i=1}^n (U_i \cup V_i)$. $\qquad \square$

I do not know if there exist measures μ in $\mathcal{M}_X$ which are not good but for which $H_\mu(X)$ acts transitively. However, with some assumptions on the clopen values sets we do obtain the converse result.

Lemma 2.14. *Let $\mu \in \mathcal{M}_X$ with X a Cantor space.*

(a) *If the action of $H_\mu(X)$ on X is transitive, minimal or topologically transitive, then for every nonempty clopen subset V of X the action of $H_{\mu_V}(V)$ on V satisfies the corresponding property.*

(b) *Assume that for every nonempty clopen subset V of X, $S(\mu_V)$ is group-like. If for every pair of clopen subsets U, V of X with $\mu(U) \leq \mu(V)$ there exist clopen subsets $U_1, ..., U_n$ of V such that*

$$(2.44) \qquad \sum_{i=1}^{n} \mu(U_i) \quad = \quad \mu(U),$$

then μ is a good measure on X.

Proof. (a) It suffices to show that for $x, y \in V$, if y is in the H_μ orbit of x, then it is in the H_{μ_V} orbit of x. It suffices to find $g \in H_\mu$ with $g(x) = y$ and $g = 1_X$ on $X \setminus V$. If $x = y$ let $g = 1_X$. Otherwise, if $f \in H_\mu$ with $f(x) = y$, we can choose a clopen set U with $x \in U$ such that U is disjoint from the clopen $f(U)$ and $U \cup f(U) \subset V$. Define g to be f on U, f^{-1} on $f(U)$ and the identity elsewhere.

(b) For $i = 1, ..., n$, $\mu_V(U_i)$ is in the group-like set $S(\mu_V)$. Since the sum $\mu(U)/\mu(V)$ is in I, it is in $S(\mu_V)$ and so $\mu(U) \in S(\mu, V)$ by (2.20). Thus, there is a clopen subset U_1 of V such that $\mu(U) = \mu(U_1)$, proving the Subset Condition. □

Theorem 2.15. *Let $\mu \in \mathcal{M}_X$ with X a Cantor space. Assume that for every nonempty clopen subset V of X, $S(\mu_V)$ is group-like. If $H_\mu(X)$ acts minimally on X, then μ is a good measure on X.*

Proof. Assume we are given a pair of clopen subsets U, V of X with $\mu(U) \leq \mu(V)$. With $H = H_\mu(X)$, $Hx \cap V \neq \emptyset$ for every $x \in U$. So for every $x \in U$ there exists $h_x \in H$ and a clopen $U_x \subset U$ such that $x \in U_x$ and $h_x(U_x) \subset V$. Choose a finite list $\{U_{x_1}, ..., U_{x_n}\}$ covering U and define for $i = 1, ..., n$

$$(2.45) \qquad \tilde{U}_i \; = \; U_{x_i} \setminus \bigcup_{j=1}^{i-1} U_{x_j} \qquad \text{and} \qquad U_i = h_{x_i}(\tilde{U}_i).$$

Clearly, each $U_i \subset V$ and (2.44) holds. It follows from Lemma 2.14(b) that μ is a good measure. □

Remark. I do not know whether the group-like assumption is redundant. That is, I do not know of any example where H_μ acts minimally, but the measure is not good. If one could prove that H_μ acts minimally $\implies$ $S(\mu)$ is group-like, then Lemma 2.14(a) would imply that $S(\mu_V)$ is group-like for every nonempty clopen V and so from the theorem, μ would be good.

A *pointed monothetic group* is a pair (Γ, g), where Γ is a Polish group and g is a topological generator of Γ, i.e. the cyclic group generated by g is dense in Γ. It follows that Γ is abelian.

A map $h : (\Gamma_1, g_1) \to (\Gamma_2, g_2)$ is a *homomorphism* if $h : \Gamma_1 \to \Gamma_2$ is a continuous group homomorphism with $h(g_1) = g_2$. If there exists a homomorphism h from (Γ_1, g_1) to (Γ_2, g_2), then it is unique. When it exists then there is a homomorphism in the other direction iff h is an isomorphism. If the domain is compact, then h is surjective.

Assume now that (Γ, g) is a compact, zero-dimensional, pointed monothetic group with Haar measure denoted μ_Γ. Haar measure is always full and it is nonatomic iff Γ is infinite. If Γ is finite with order n, then (Γ, g) is isomorphic to $(\mathbb{Z}_n, 1)$, the cyclic group of integers mod n with 1 the congruence class of the number 1. If Γ is infinite, then it is topologically homogeneous and not discrete

and so it is a Cantor space. We will call (Γ, g) a *pointed Cantor group* when it is an infinite, compact, zero-dimensional pointed monothetic group.

If (Γ, g) is a pointed Cantor group and G is a clopen subgroup of Γ, then the set Γ/G of cosets is a partition of Γ and so G has finite index. The quotient $(\Gamma/G, g + G)$ is isomorphic to $(\mathbb{Z}_n, 1)$, where n is the index of G, denoted hereafter *ind* G. Conversely, if there is a homomorphism from (Γ, g) to $(\mathbb{Z}_n, 1)$, then the kernel is a clopen subgroup of index n.

By starting with an ultrametric on the Cantor space Γ and averaging using μ_Γ, we can obtain an invariant ultrametric d_Γ on Γ, i.e. if $g_1, g_2, g_3 \in \Gamma$, then

(2.46)
$$d_\Gamma(g_1, g_3) \leq max(d_\Gamma(g_1, g_2), d_\Gamma(g_2, g_3))$$
$$\text{and} \quad d_\Gamma(g_1, g_2) = d_\Gamma(g_1 + g_3, g_2 + g_3).$$

If $\epsilon > 0$ then the d_Γ ball of radius ϵ about 0 is a clopen subgroup. Hence, the family $\mathcal{G}_\Gamma$ of clopen subgroups of Γ is a countable base for the neighborhoods of 0 in Γ and the collection of cosets of such clopen subgroups is a countable base for the topology of Γ. Clearly, $G_1, G_2 \in \mathcal{G}_\Gamma$ implies $G_1 \cap G_2 \in \mathcal{G}_\Gamma$ with

(2.47)
$$lcm(ind\ G_1, ind\ G_2) \mid ind\ G_1 \cap G_2 \mid ind\ G_1 \cdot ind\ G_2.$$

Define the set of positive integers

(2.48)
$$D_\Gamma \quad =_{def} \quad \{ind\ G : G \in \mathcal{G}_\Gamma\}.$$

We call a set D of positive integers a *divisibility set* if

(2.49) $\quad m|n \quad \text{and} \quad n \in D \implies m \in D \quad \text{and} \quad m, n \in D \implies lcm(m, n) \in D.$

Clearly, if (Γ, g) is a pointed Cantor group, then D_Γ is a divisibility set. In general, for $S \subset I$ with $0, 1 \in S$, if S is group-like, then $Div(S) \cap \mathbb{Z}_+$ is a divisibility set and by Lemma 2.2 $Rec(S)$ is a divisibility set.

Let D be an infinite divisibility set. The D-indexed family $\{(\mathbb{Z}_n, 1) : n \in D\}$ with the canonical homomorphism from $(\mathbb{Z}_n, 1)$ to $(\mathbb{Z}_m, 1)$ when $m|n$ has as its inverse limit a pointed Cantor group denoted $(\Gamma_D, 1)$. It is called the *adding machine associated with D*. By counting D and letting m_k be the least common multiple of the first k members of D we can obtain a cofinal, increasing sequence $\{m_k : k = 0, 1...\}$ in D, $m_k | m_{k+1}$ for $k = 0, 1, ...$ Then $(\Gamma_D, 1)$ is isomorphic to the inverse limit of the sequence of projections $\{(\mathbb{Z}_{m_{k+1}}, 1) \to (\mathbb{Z}_{m_k}, 1) : k = 0, 1, ...\}$.

If $D = D_\Gamma$ for any pointed Cantor group (Γ, g), then the homomorphisms $(\Gamma, g) \to (\Gamma/G, g + G) \cong (\mathbb{Z}_n, 1)$, where $G \in \mathcal{G}_\Gamma$ and *ind* $G = n$, induce an isomorphism from (Γ, g) onto the adding machine for D_Γ.

If S is a group-like subset of I with $0, 1 \in S$ such that $Rec(S)$ is infinite, or, equivalently by Lemma 2.2, $\mathbb{Q} \cap S$ is dense in I, then we call $(\Gamma_{Rec(S)}, 1)$ the *adding machine associated with S*.

Theorem 2.16. *Let (Γ, g) be a pointed Cantor group. If μ_Γ is Haar measure on Γ, then μ_Γ is a good measure on Γ with*

(2.50)
$$S(\mu_\Gamma) = \{k/n : k = 0, 1, ..., n \text{ and } n \in D_\Gamma\}.$$

Conversely, if μ is a good measure on a Cantor space X and $S(\mu) \subset \mathbb{Q}$, then with $\Gamma = \Gamma_{Rec(S)}$ there is a homeomorphism $h : X \to \Gamma$ such that $h_*\mu = \mu_\Gamma$. That is, any good measure with clopen values set contained in the rationals is homeomorphic to the Haar measure on a pointed Cantor group. Furthermore, the measure μ uniquely determines the group up to isomorphism of pointed Cantor groups.

Proof. The cosets of clopen subgroups of Γ form a basis for topology of Γ. Hence, if U, V are clopens in Γ there exists $G \in \mathcal{G}_\Gamma$ so that both U and V are unions of cosets of G. That is, if $n = ind\, G$ and π is the associated homomorphism onto $\Gamma/G \cong \mathbb{Z}_n$, then there exist finite $R, S \subset \mathbb{Z}_n$ such that $U = \pi^{-1}(R)$ and $V = \pi^{-1}(S)$. Because π maps μ_Γ onto the uniform measure on $\mathbb{Z}_n$ we have

$$(2.51) \qquad\qquad \mu_\Gamma(U) \; = \; \#R/n \quad \text{and} \quad \mu_\Gamma(V) \; = \; \#S/n,$$

where $\#R$ is the cardinality of R. From this equation (2.50) is clear. Also if $\mu(U) \leq \mu(V)$, then there exists $R_1 \subset S$ such that $\#R = \#R_1$ and so $\pi^{-1}(R_1)$ is a clopen subset of V with the same measure as U. This verifies the Subset Condition and so μ_Γ is good.

Conversely, if μ is a good measure and $S(\mu) = S(\mu) \cap \mathbb{Q}$, then by Lemma 2.2 and (2.50) the good measures μ and μ_Γ with $\Gamma = \Gamma_{Rec(S)}$ have the same clopen values sets. In particular, since $S(\mu)$ is dense in I, Lemaa 2.2 implies that $Rec(S)$ is infinite $S(\mu) = S(\mu) \cap \mathbb{Q}$. By Theorem 2.9(a) they are homeomorphic measures.

Finally, if two pointed Cantor groups have homeomorphic Haar measures, then they have a common clopen values set S and so each is isomorphic to $(\Gamma_{Rec(S)}, 1)$ in the category of pointed Cantor groups. $\qquad\square$

Theorem 2.17. *Let (Γ_1, g_1) and (Γ_2, g_2) be pointed Cantor groups with Haar measures μ_{Γ_1} and μ_{Γ_2}, respectively. The following conditions are equivalent:*

(1) $S(\mu_{\Gamma_1}) \supset S(\mu_{\Gamma_2})$.

(2) $D_{\Gamma_1} \supset D_{\Gamma_2}$.

(3) *There exists a continuous map $f : \Gamma_1 \to \Gamma_2$ such that $f_*\mu_{\Gamma_1} = \mu_{\Gamma_2}$.*

(4) *There exists a continuous, almost one-to-one surjection $f : \Gamma_1 \to \Gamma_2$ such that $f_*\mu_{\Gamma_1} = \mu_{\Gamma_2}$.*

(5) *There exists $h : (\Gamma_1, g_1) \to (\Gamma_2, g_2)$ a homomorphism of pointed Cantor groups.*

Any homomorphism $h : (\Gamma_1, g_1) \to (\Gamma_2, g_2)$ of pointed Cantor groups is surjective and satisfies $h_*\mu_{\Gamma_1} = \mu_{\Gamma_2}$.

Proof. $(1) \Leftrightarrow (3) \Leftrightarrow (4)$ follow from Theorem 2.9(b) because the Haar measures on pointed Cantor groups are good.

$(1) \Leftrightarrow (2)$ By (2.50) D_Γ determines $S(\mu_\Gamma)$. On the other hand, by (2.50) and (2.12), $D_\Gamma = Rec(S(\mu_\Gamma))$ and so D_Γ is determined by $S(\mu_\Gamma)$.

$(2) \Leftrightarrow (5)$ If $h : (\Gamma_1, g_1) \to (\Gamma_2, g_2)$ is a homomorphism, then $G \in \mathcal{G}_{\Gamma_1}$ implies $h^{-1}(G) \in \mathcal{G}_{\Gamma_2}$ with $ind\, G = ind\, h^{-1}(G)$. Consequently, $D_{\Gamma_1} \supset D_{\Gamma_2}$. On the other hand, from the adding machine construction expressing $(\Gamma, 1)$ as the inverse limit of the finite groups $\{(\Gamma/G, 1 + G) : G \in \mathcal{G}_\Gamma\}$, it is easy to see that the inclusion of (2) implies that the homomorphism $h : (\Gamma_1, g_1) \to (\Gamma_2, g_2)$ exists.

If $h : (\Gamma_1, g_1) \to (\Gamma_2, g_2)$ is a homomorphism, then because h is surjective and μ_{Γ_1} is translation invariant it follows that $h_*\mu_{\Gamma_1}$ is translation invariant and so is the unique translation invariant measure μ_{Γ_2} on Γ_2. $\qquad\square$

We will call (Ξ, g) the pointed Cantor group with D_Ξ the entire set of positive integers, i.e. the pointed Cantor group with clopen subgroups of every positive index. We call (Ξ, g) the *universal adding machine.* By (2.50) and Theorem 2.16, the Haar measure μ_Ξ is characterized up to homeomorphism as the unique good measure with clopen values set

$$(2.52) \qquad\qquad\qquad S(\mu_\Xi) \;\; = \;\; \mathbb{Q} \cap I.$$

For each prime p the *ring of p-adic integers* Γ_p is the inverse limit of the sequence of projections $\mathbb{Z}_{p^{k+1}} \to \mathbb{Z}_{p^k}$. The generator g_p is the sequence of congruence classes of 1. The pointed additive group (Γ_p, g_p) is a pointed Cantor group and we denote by μ_p the associated Haar measure. Clearly,

$$(2.53) \qquad D(\Gamma_p) \quad = \quad \{p^i : i = 0, 1, 2, ...\}$$

and so from (2.50)

$$(2.54) \qquad S(\Gamma_p) \quad = \quad \{k/p^i : 0 \le k \le p^i \text{ and } i = 0, 1, 2, ...\}.$$

The topological group Ξ is the product of copies of Γ_p as the index p varies over all primes.

If $[n] = \{1, ..., n\}$ with $n > 1$ and $w : [n] \to I \setminus \{0, 1\}$ is a positive distribution, i.e. $\sum w_i = 1$, then the *associated Bernoulli measures*, denoted $\beta(w_1, ..., w_n)$, on $[n]^{\mathbb{Z}_+}$ and $[n]^{\mathbb{Z}}$ are the unique probability measures such that the projections to different factors are independent and the projection to each factor induces the distribution w on $[n]$. Any bijection between $\mathbb{Z}_+$ and $\mathbb{Z}$ induces a homeomorphism between these two. The Bernoulli measures are full and nonatomic and on $X = [n]^{\mathbb{Z}}$ the shift homeomorphism s defined by $s(x)_i = x_{i+1}$ preserves the Bernoulli measure.

By using the base p expansion of integers it is easy to obtain a homeomorphism from Γ_p to $[p]^{\mathbb{Z}_+}$ which maps μ_p to the uniform Bernoulli measure $\beta(1/p, ..., 1/p)$. In particular, on $[3]^{\mathbb{Z}_+}$ the Bernoulli measure $\beta(1/3, 1/3, 1/3)$ is a good measure with clopen values set $\{k/3^i : 0 \le k \le 3^i \text{ and } i = 0, 1, 2, ...\}$. We conclude this section with an example which shows how close a bad measure can get to a good one. Compare the results to those of Theorem 2.15.

Theorem 2.18. *If on $X = [2]^{\mathbb{Z}_+}$ the measure μ is the Bernoulli measure $\beta(1/3, 2/3)$, then $\mu \in \mathcal{M}_X$ with $S(\mu) = S(\mu_3) = \{k/3^i : 0 \le k \le 3^i \text{ and } i = 0, 1, 2, ...\}$. Furthermore, for every nonempty clopen subset V of X, $S(\mu_V)$ is group-like and the action of the group $H_\mu(X)$ on X is topologically transitive. On the other hand, if e is the point in X with $e_i = 1$ for all $i \in \mathbb{Z}_+$, then $f(e) = e$ for all $f \in H_\mu(X)$, i.e. e is a fixed point for the action of the group. In particular, μ is not a good measure on X.*

Proof. The computation of $S(\mu)$ is given in Proposition 1.7 of Akin [1999]. The computation of $S(\mu_V)$ for general clopen V uses similar methods but is rather elaborate and so we will omit it here. The shift map s on $[2]^{\mathbb{Z}}$ is topologically transitive and preserves the Bernoulli measure. Since the two versions of the Bernoulli measure are homeomorphic, it certainly follows that the action of $H_\mu(X)$ is topologically transitive. In fact, one can show that if for $x \in X$ both $x_i = 1$ and $x_i = 2$ for infinitely many i, then the H_μ orbit of x is dense.

On the other hand, let f be a homeomorphism on X with $f(e) \ne e$. There exists a clopen set V with $e \in V$ and $f(V)$ disjoint from V. There exists some positive integer N and disjoint subsets F_1, F_2 of $[2]^{[N]}$ such that $V = \pi^{-1}(F_1)$ and $f(V) = \pi^{-1}(F_2)$, where $\pi : [2]^{\mathbb{Z}_+} \to [2]^{[N]}$ is the projection. Furthermore, $(1, ..., 1) \in F_1$ and so $(1, ..., 1) \notin F_2$. It follows that $\mu(V) = m_1/3^N$ with m_1 odd while $\mu(f(V)) = m_2/3^N$ with m_2 even. Thus, f is not measure preserving.

By Theorem 2.13, H_μ acts transitively on X if μ is good. So in this case, μ is not good. $\qquad \square$

Remark. In Proposition 1.7 of Akin [1999] we showed directly that $\beta(1/3, 2/3)$ is not homeomorphic to $\beta(1/3, 1/3, 1/3)$. Since the clopen values set is a complete invariant for good measures, it follows that they cannot both be good. Since $\beta(1/3, 1/3, 1/3)$ is good, we see another reason why $\beta(1/3, 2/3)$ cannot be good.

3. The field-like case

Let $\mathbb{R}^*$ denote the multiplicative group of positive reals, with $\mathbb{Q}^* = \mathbb{R}^* \cap \mathbb{Q}$ and $\mathbb{Z}^* = \mathbb{R}^* \cap \mathbb{Z}$, so that $\mathbb{Q}^*$ is a subgroup and $\mathbb{Z}^*$ is at least closed under multiplication.

Recall that if G is an additive subgroup of $\mathbb{R}$, then

$$(3.1) \qquad Div(G) \quad =_{def} \quad \{\alpha \in \mathbb{R}^* : \alpha G = G\}$$

is a multiplicative subgroup of $\mathbb{R}^*$. When S is a group-like subset with $0, 1 \in S$, we denote by $Div(S)$ the multiplicative group $Div(G)$, where $G = G[S] = S + \mathbb{Z}$. Clearly,

$$(3.2) \qquad S \text{ is } \mathbb{Q}\text{-like} \quad \Longleftrightarrow \quad \mathbb{Q}^* \subset Div(S) \quad \Longleftrightarrow \quad \mathbb{Z}^* \subset Div(S).$$

Lemma 3.1. *Let μ be a good measure on a Cantor space X and let V be a nonempty clopen subset of X.*

$$(3.3) \qquad Div(S(\mu_V)) \quad = \quad Div(S(\mu)).$$

Furthermore, the following conditions are equivalent:

1. $\mu(V) \in Div(S(\mu))$.
2. $S(\mu_V) = S(\mu)$.
3. *There exists a homeomorphism $h : V \to X$ such that*

$$(3.4) \qquad h_*\mu_V \quad = \quad \mu.$$

If $\leq$ is an ordering on X which is adapted to μ, then the above conditions imply that there is a unique order isomorphism $h : (V, \leq) \to (X, \leq)$ such that (3.4) holds.

Proof. Equivalence (2.6) implies that

$$(3.5) \qquad \alpha \in Div(S) \qquad \Longleftrightarrow \qquad S = (\alpha^{-1}S) \cap I.$$

By (2.21) and (2.5) the group generated by $S(\mu_V)$ equals $(\mu(V))^{-1}(S(\mu) + \mathbb{Z})$. Hence, $\alpha \in Div(S(\mu))$ iff $\alpha \in Div(S(\mu_V))$, proving (3.3).

$(1) \Leftrightarrow (2)$ follows from (2.21) and (3.5).

$(2) \Leftrightarrow (3)$ follows from Theorem 2.9(a).

When the order on X is adapted to μ, then the order on V is adapted to μ_V by Theorem 2.6(c). If condition (2) holds, then two two measures have the same special clopen values sets and so the unique order isomorphism h exists by Theorem 1.6(b). $\qquad\square$

Proposition 3.2. *Let μ be a good measure on a Cantor space X. The multiplicative group $Div(S(\mu))$ is nontrivial iff there is a basis of clopen sets U of X such that μ_U on U is homeomorphic to μ on X.*

Proof. If $\alpha \in Div(S(\mu)) \setminus \{1\}$, then 0 is a limit point of the subgroup $\{\alpha^n : n \in \mathbb{Z}\}$. If $x \in X, V \subset X$ is clopen and $\alpha \in Div(S(\mu))$ with $\alpha \leq \mu(V)$, then by Proposition 2.11(b) there exists a clopen $U \subset V$ with $x \in U$ and $\mu(U) = \alpha$. Hence,

$$(3.6) \qquad \mathcal{U} \quad =_{def} \quad \{U \subset X : U \text{ is clopen and } \mu(U) \in Div(S(\mu))\}$$

is a basis for X. By Lemma 3.1 a clopen U is in $\mathcal{U}$ iff μ_U is homeomorphic to μ. Conversely, Lemma 3.1 implies that $Div(S(\mu))$ is nontrivial if there is any proper clopen subset U such that μ_U is homeomorphic to μ. $\qquad\square$

Proposition 3.3. *For μ a good measure on a Cantor space X the following conditions are equivalent:*

(1) $S(\mu)$ *is* $\mathbb{Q}$*-like.*
(2) $\mathbb{Q} \cap I \subset S(\mu)$ *and for every nonempty clopen subset U of X with $\mu(U)$ rational, μ_U on U is homeomorphic to μ on X.*
(3) $\mathbb{Q} \cap I \subset S(\mu)$ *and for every nonempty clopen subset U of X with $\mu(U)$ rational, $S(\mu_U) = S(\mu)$.*

Proof. By Lemma 3.1 each of these conditions says exactly that $\mathbb{Q}^* \cap I \subset Div(S(\mu))$. $\qquad\square$

Theorem 3.4. *Let X be a Cantor space and $\mu \in \mathcal{M}_X$. The following conditions on μ are equivalent:*

(1) μ *is a good measure on X and $S(\mu)$ is field-like.*
(2) μ *is a good measure on X and for every nonempty clopen subset U of X, μ_U on U is homeomorphic to μ on X.*
(3) $S(\mu)$ *is group-like and for every nonempty clopen subset U of X, $S(\mu_U) = S(\mu)$.*
(4) *For every nonempty clopen subset U of X, $S(\mu_U) = S(\mu)$ and, in addition,*

$$(3.7) \qquad\qquad (2S(\mu)) \cap I \ \subset \ S(\mu).$$

Proof. (1) $\Rightarrow$ (2) If $S(\mu)$ is field-like, then for every nonempty clopen U, $\mu(U) \in Div(S(\mu))$. Apply Lemma 3.1.

(2) $\Rightarrow$ (3) Proposition 2.4 implies $S(\mu)$ is group-like and Proposition 1.5 implies that $S(\mu_U) = S(\mu)$.

(3) $\Rightarrow$ (4) $S(\mu)$ group-like implies (3.7).

(4) $\Rightarrow$ (1) We first show that if for every nonempty clopen subset U of X $S(\mu_U) = S(\mu)$, then conditions (i)-(iv) of Lemma 2.1(f) hold.

If $\mu(U) = \alpha$, then $\mu(X \setminus U) = 1 - \alpha$. So condition (ii) holds for any $S(\mu)$.

If $\alpha, \beta \in S(\mu)$, then there exists a clopen V with $\mu(V) = \alpha$. If $\alpha > 0$, then $S(\mu_V) = S(\mu)$ and so there exists a clopen $U \subset V$ with $\mu_V(U) = \beta$ and so $\mu(U) = \alpha\beta$. In particular, $S(\mu)$ is closed under multiplication.

Given a positive integer n let $\{V_1, ..., V_n\}$ be a partition of cardinality n. As above we can find for $i = 1, ..., n$ clopens $U_i \subset V_i$ such that $\mu(U_i)$ is the product $\mu(V_1)...\mu(V_n)$. Let U be the clopen $U_1 \cup ... \cup U_n$. Clearly, $\{U_1, ..., U_n\}$ is a partition of U with $\mu_U(U_i) = 1/n$ for $i = 1, ..., n$. Hence, $k/n \in S(\mu_U) = S(\mu)$ for $k = 0, ..., n$. So condition (i) holds.

Applying this construction with $n = 2$ we can find two disjoint nonempty clopens U_1, U_2 with $\mu(U_1)$ and $\mu(U_2)$ a common positive number γ and so with $U = U_1 \cup U_2$ $\mu(U) = 2\gamma$. If $\alpha, \beta \in S(\mu)$, then there exist clopens $W_1 \subset U_1$ and $W_2 \subset U_2$ such that $\mu_{U_1}(W_1) = \alpha$ and $\mu_{U_2}(W_2) = \beta$. Let $W = W_1 \cup W_2$. Clearly,

$$(3.8) \qquad \mu_U(W) \ = \ (\alpha + \beta)/2 \qquad \text{and} \qquad \mu_W(W_1) \ = \ \alpha/(\alpha + \beta).$$

Since $S(\mu_U) = S(\mu) = S(\mu_W)$ this verifies conditions (iii) and (iv) of Lemma 2.1(f).

Assumption (3.7) implies that condition (v) holds as well. So by Lemma 2.1(f) $S(\mu)$ is field-like. It remains to verify the Subset Condition.

Given clopens U, V with $\mu(U) = \alpha < \beta = \mu(V)$ then $S(\mu)$ field-like implies that $\alpha/\beta \in S(\mu) = S(\mu_V)$ and so there exists a clopen $U_1 \subset V$ such that $\mu_V(U_1) = \alpha/\beta$. Hence, $\mu(U_1) = \alpha = \mu(U)$.

Since the Subset Condition holds, μ is good on X. $\qquad\square$

Remark. While we need condition (3.7) for the proof, it may well be redundant. We have no examples which suggest otherwise.

Corollary 3.5. *Let $(X, \leq)$ be an ordered Cantor space and $\mu \in \mathcal{M}_X$. If for every nonempty clopen subset V of X*

$$(3.9) \qquad\qquad S(\mu_V) \quad = \quad S(\mu) \quad = \quad \tilde{S}(\mu),$$

then μ is good on X and $S(\mu)$ is field-like.

Proof. By definition μ is adapted to $(X, \leq)$ and so by Theorem 2.6(b) μ is good on X. Thus, condition (3) of Theorem 3.4 holds and (3) $\Rightarrow$ (1) of the theorem yields the result. $\qquad\square$

Theorem 3.6. *Let μ be a good measure on a Cantor space X and let μ_1 be a full measure on a zero-dimensional space X_1. The product space $X_1 \times X$ is a Cantor space and the product measure $\mu_1 \times \mu$ is full and nonatomic. If, in addition, we have*

$$(3.10) \qquad\qquad S(\mu_1) \setminus \{0\} \quad \subset \quad Div(S(\mu)),$$

then there is a homeomorphism $h : X \to X_1 \times X$ such that

$$(3.11) \qquad\qquad h_*\mu \quad = \quad \mu_1 \times \mu.$$

In particular, $\mu_1 \times \mu$ is then a good measure on $X_1 \times X$ with

$$(3.12) \qquad\qquad S(\mu) \quad = \quad S(\mu_1 \times \mu).$$

Proof. Observe that $X_1 \times X$ is a zero-dimensional space and it is perfect because X is. Thus, the product is a Cantor space. Clearly, the product measure is full and it is nonatomic because μ is. To obtain the homeomorphism h it suffices by Theorem 2.9(a) to prove directly that the product measure is good and that (3.12) holds.

Since the projection of $X_1 \times X$ to X maps $\mu_1 \times \mu$ to μ, it follows from (1.23) that $S(\mu) \subset S(\mu_1 \times \mu)$. Any nonempty clopen W in the product is a finite disjoint union of products $U_i \times V_i$ with U_i clopen in X_1 and V_i clopen in X with $\alpha_i = \mu_1(U_i) > 0$ and $\beta_i = \mu(V_i) > 0$. Each $\beta_i \in S(\mu)$ and by assumption (3.10) each $\alpha_i \in Div(S(\mu))$. Because $S(\mu)$ is group-like, $\mu(W) = \sum_i \alpha_i \beta_i \in S(\mu)$. Hence, (3.12) holds.

Because μ is good (2.17) implies that $S(\mu, V_i) \supset S(\mu) \cap [0, \beta_i]$ and so

$$(3.13) \qquad S(\mu_1 \times \mu, U_i \times V_i) \quad \supset \quad \alpha_i S(\mu, V_i) \quad \supset \quad S(\mu) \cap [0, \alpha_i \beta_i],$$

since α_i is a divisor of $S(\mu) + \mathbb{Z}$. If $\gamma < \sum_i \alpha_i \beta_i$, then there is a positive integer r such that

$$(3.14) \qquad\qquad \sum_{i<r} \alpha_i \beta_i \quad < \quad \gamma \quad \leq \quad \sum_{i<r} \alpha_i \beta_i + \alpha_r \beta_r.$$

If, in addition, $\gamma \in S(\mu)$, then by equations (3.14) and (3.12), $\gamma - \sum_{i<r} \alpha_i \beta_i \in S(\mu_1 \times \mu, U_r \times V_r)$. It follows that $\gamma \in S(\mu_1 \times \mu, W)$. Applying (2.16) with $V = W$ we obtain (2.17) for every clopen subset W. It follows that $\mu_1 \times \mu$ satisfies the Subset Condition. $\qquad\square$

Corollary 3.7. *Assume that μ is a good measure on a Cantor space X with $S(\mu)$ field-like. Let Y be the product of a finite or countably infinite number of copies of X and let ν be the product measure on Y obtained with μ on each factor. The measure ν is good on the Cantor space Y and there exists a homeomorphism $h : X \to Y$ such that*

$$(3.15) \qquad h_* \mu \quad = \quad \nu.$$

Proof. For $\mu \times \mu$ on $X \times X$ the result is Theorem 3.6 with $\mu_1 = \mu$. For any finite number of copies the result follows by induction using Theorem 3.6 again.

If Y is the product of copies of X indexed by a countably infinite set J and $U \subset V$ are clopen subsets of Y, then there exist a clopen subset $\tilde{U} \subset \tilde{V}$ of a subproduct indexed by some finite subset F of J such that U and V are the preimages of $\tilde{U}$ and $\tilde{V}$, respectively, via the projection map. It then follows from the finite product result that $S(\nu_V) = S(\mu)$. By $(3) \Rightarrow (1)$ of Theorem 3.4 it follows that ν is good on Y. That $h_* \mu = \nu$ then follows from Theorem 2.9(a). $\qquad\square$

4. Automorphisms of good measures

In constructing automorphisms of a good measure it is useful to use an adapted ordering. As an illustration we have

Proposition 4.1. *Let $(X, \leq)$ be an ordered Cantor space with $\mu \in \mathcal{M}_X$. Assume that*

$$(4.1) \qquad \alpha \in \tilde{S}(\mu) \qquad \Longrightarrow \qquad 1 - \alpha \in \tilde{S}(\mu),$$

e.g. this holds if μ is adapted to $(X, \leq)$. There exists a unique $f \in H_\mu(X)$ which is order-reversing, i.e.

$$(4.2) \qquad x \leq y \qquad \Longrightarrow \qquad f(x) \geq f(y).$$

Proof. Let $\leq'$ denote the reverse order on X. Since μ is nonatomic it is clear that α is in the special clopen values set for μ on $(X, \leq)$ iff $1 - \alpha$ is in the special clopen values set for μ on $(X, \leq')$. By assumption these two sets are the same. So Theorem 1.6(b) implies there is a unique measure preserving order isomorphism from $(X, \leq)$ to $(X, \leq')$. $\qquad\square$

Remarks. (a) In general, $\alpha = \mu(U)$ implies that $1 - \alpha = \mu(X \setminus U)$ and so it is always true that $\alpha \in S(\mu)$ implies $1 - \alpha \in S(\mu)$. Hence, $S(\mu) = \tilde{S}(\mu)$ implies (4.1).

(b) If we let F'_μ denote the CDF of μ with respect to the reverse order $\leq'$, it is clear that

$$(4.3) \qquad F'_\mu \quad = \quad 1 - F_\mu.$$

So let us begin with $(X, \leq)$ an ordered Cantor space and a measure μ adapted to $(X, \leq)$, i.e. $\tilde{S}(\mu) = S(\mu)$. Recall from Proposition 1.3 that the CDF $F_\mu : (X, \leq) \to (I, \leq)$ is an almost one-to-one, continuous surjection with $F_\mu(I \setminus Inj_{F_\mu}) = S(\mu) \setminus \{0, 1\}$. For $t \in S(\mu) \setminus \{0, 1\}$ we let $t^- < t^+$ denote the endpoint pair which is mapped to t by F_μ. For 0, we will write $0^- = 0^+ = m$ and $1^- = 1^+ = M$, the minimum and maximum point, respectively. We will call a closed interval $J = [t_1, t_2]$ of $\mathbb{R}$ a μ *interval* if

$$(4.4) \qquad 0 \leq t_1 < t_2 \leq 1 \qquad \text{and} \qquad t_1, t_2 \in S(\mu),$$

in which case we let

$$(4.5) \qquad \hat{J} \quad =_{def} \quad [t_1^+, t_2^-],$$

so that $\hat{J}$ is the unique clopen interval which is mapped by F_μ onto J. Clearly,

$$(4.6) \qquad \begin{aligned} F_\mu^{-1}(J) &= \hat{J} \cup \{t_1^-, t_2^+\} &= [t_1^-, t_2^+], \\ F_\mu^{-1}(J^\circ) &= \hat{J} \setminus \{t_1^+, t_2^-\} &= (t_1^+, t_2^-), \end{aligned}$$

where J° is the interior of J in $\mathbb{R}$.

For $\alpha \in \mathbb{R}$ define the *translation map* H_α and the *antitranslation map* H_α^- on $\mathbb{R}$ by

$$(4.7) \qquad H_\alpha(t) \;=\; t + \alpha \qquad \text{and} \qquad H_\alpha^-(t) \;=\; -t + \alpha.$$

For a subinterval $J \subset \mathbb{R}$ let $|J|$ denote its length, which equals its Lebesgue measure $\lambda(J)$. If J_1, J_2 are closed subintervals of $\mathbb{R}$ with $|J_1| = |J_2| > 0$, then there is a unique translation map H_{21}^+ and a unique antitranslation map H_{21}^- such that

$$(4.8) \qquad H_{21}^+(J_1) \quad = \quad J_2 \quad = \quad H_{21}^-(J_1).$$

Lemma 4.2. *Assume that $(X, \leq)$ is an ordered Cantor space and μ is a measure adapted to $(X, \leq)$. Let J_1 and J_2 be $S(\mu)$ intervals with $|J_1| = |J_2|$. There exist unique measure preserving homeomorphisms $h_{21}^+, h_{21}^- : \hat{J}_1 \to \hat{J}_2$ such that h_{21}^+ is order-preserving and h_{21}^- is order-reversing. Furthermore, these are the unique continuous maps such that*

$$(4.9) \qquad H_{21}^\pm \circ F_\mu \quad = \quad F_\mu \circ h_{21}^\pm.$$

Proof. By Theorem 2.6(c) $\mu_{\hat{J}_k}$ is a measure adapted to $(\hat{J}_k, \leq)$ $(k = 1, 2)$. By Theorem 2.6(a) they are good and so by (2.21) $\tilde{S}(\mu_{\hat{J}_1}) = \tilde{S}(\mu_{\hat{J}_2})$. By Theorem 1.6 there is a unique order isomorphism h_{21}^+ from $\mu_{\hat{J}_1}$ to $\mu_{\hat{J}_2}$ and it is the unique continuous map such that $F_2 \circ h_{21}^+ = F_1$, where F_k is the CDF of $\mu_{\hat{J}_k}$ for $k = 1, 2$. The restriction of F_μ to $\hat{J}_k$ is F_k composed with the order preserving linear map from I to J_k $(k = 1, 2)$ and so the characterization (4.7) for h_{21}^+ follows. For h_{21}^- apply Proposition 4.1 and the Remarks thereafter. $\qquad\square$

By the *circle* we will mean the compact quotient group $\mathbb{R}/\mathbb{Z}$ with the projection homomorphism $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. π restricts to a surjection $\pi : I \to \mathbb{R}/\mathbb{Z}$ which yields the circle as the space I with 0 and 1 identified. We will call an element $\alpha \in \mathbb{R}/\mathbb{Z}$ rational (or irrational) if it is the projection under π of a rational (resp. an irrational) real number.

If F_μ is the CDF of the measure μ we define the surjection F_μ^π and the subset $S^\pi(\mu)$ by

$$(4.10) \qquad \begin{aligned} F_\mu^\pi &=_{def} \quad \pi \circ F_\mu : X \to \mathbb{R}/\mathbb{Z}, \\ S^\pi(\mu) \quad =_{def} \quad F_\mu^\pi(S(\mu)) &= F_\mu^\pi(S(\mu) + \mathbb{Z}) \quad \subset \quad \mathbb{R}/\mathbb{Z}. \end{aligned}$$

For $\alpha \in \mathbb{R}/\mathbb{Z}$ define the *translation map*

$$(4.11) \qquad H_\alpha^\pi(t) \quad = \quad t + \alpha,$$

with addition in $\mathbb{R}/\mathbb{Z}$.

Theorem 4.3. *Assume that $(X, \leq)$ is an ordered Cantor space and μ is a measure adapted to $(X, \leq)$. $S^{\pi}(\mu)$ is a countable, dense subgroup of $\mathbb{R}/\mathbb{Z}$. For each $\alpha \in S^{\pi}(\mu)$ there is a unique $h_{\alpha} \in H_{\mu}(X)$ such that*

$$(4.12) \qquad\qquad F_{\mu}^{\pi} \circ h_{\alpha} \quad = \quad H_{\alpha}^{\pi} \circ F_{\mu}^{\pi}.$$

Furthermore, the map $h_{\#} : S^{\pi}(\mu) \to H_{\mu}(X)$ defined by $\alpha \mapsto h_{\alpha}$ is a discontinuous group homomorphism.

Proof. As noted in (4.10), $S^{\pi}(\mu)$ is the image of the group $S(\mu) + \mathbb{Z}$ under the homomorphism F_{μ}^{π} and so it is a subgroup of the circle.

For $\alpha \in S(\mu) \setminus \{0, 1\}$ consider the partitions $\{[m, (1-\alpha)^-], [(1-\alpha)^+, M]\}$ and $\{[\alpha^+, M], [m, \alpha^-]\}$ of X. We use Lemma 4.2 to get two translation maps whose union is $h_{\alpha} \in H_{\mu}(X)$ which satisfies

$$(4.13) \quad h_{\alpha}([m, (1-\alpha)^-]) \;=\; [\alpha^+, M] \qquad \text{and} \qquad h_{\alpha}([(1-\alpha)^+, M]) \;=\; [m, \alpha^-],$$

and which is order preserving on each piece. Equation (4.12) follows from (4.9).

We clearly have $H_{\alpha}^{\pi} \circ H_{\beta}^{\pi} = H_{\alpha+\beta}^{\pi}$ for $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$. It follows from uniqueness of the lifting that $h_{\#}$ is a homomorphism. Since the homomorphism cannot be extended to the entire circle, it is not continuous. $\qquad\square$

We now consider the dynamic character of these maps. A homeomorphism f on a space X is called *uniquely ergodic* if it has a unique invariant measure. In the proposition below we collect the well-known results that we will need about such automorphisms.

First, we recall that if $G : X_1 \to X_2$ is a continuous map, then it is called *almost one-to-one* if $Inj_G = \{x \in X_1 : G^{-1}(G(x))$ is a singleton$\}$ is dense in X_1. By compactness $\{y \in X_2 : d(G^{-1}(y)) < \epsilon\}$ is open for any positive ϵ. Hence, $G(Inj_G)$ and its preimage Inj_G are G_{δ} sets. By Lemma 1.1(d) an almost one-to-one map is almost open.

Proposition 4.4. (a) *Let μ be a full measure on a space X. If $f \in H_{\mu}(X)$ is uniquely ergodic, then f is minimal on X. That is, for every $x \in X$ the orbit $\{f^n(x) : n \in \mathbb{Z}\}$ is dense in X. Furthermore, μ is nonatomic or X is finite.*

(b) *Let (Γ, g) be a compact pointed monothetic group. The translation map h_g on Γ defined by $h_g(x) = g + x$ is uniquely ergodic with Haar measure μ_{Γ} the unique invariant measure.*

(c) *For $k = 1, 2$ let μ_k be a measure on a space X_k and let $G : X_1 \to X_2$ be a continuous map such that*

$$(4.14) \qquad\qquad G_* \mu_1 \quad = \quad \mu_2.$$

If μ_2 is nonatomic, then μ_1 is nonatomic. If μ_2 is full, then G is surjective. In any case,

$$(4.15) \qquad \mu_2(X_2 \setminus G(Inj_G)) \;=\; 0 \qquad \Longleftrightarrow \qquad \mu_1(X_1 \setminus Inj_G) \;=\; 0.$$

We will say that G induces a measure isomorphism when these conditions hold. *If μ_2 is full and G induces a measure isomorphism, then G is almost one-to-one iff μ_1 is full.*

If μ_2 is nonatomic and $X_2 \setminus G(Inj_G)$ is countable, then G induces a measure isomorphism.

(d) *For $k = 1, 2$ let μ_k be a measure on a space X_k and $f_k \in H_{\mu_k}(X)$ and let $G : X_1 \to X_2$ be a continuous map such that*

$$(4.16) \qquad G \circ f_1 \quad = \quad f_2 \circ G.$$

If f_2 is uniquely ergodic, then $G_*\mu_1 = \mu_2$. If, in addition, μ_2 is full and nonatomic and G induces a measure isomorphism, then G is an almost one-to-one surjection, μ_1 is full and nonatomic and f_1 is uniquely ergodic.

Proof. (a) The restriction of f to any minimal subset of X admits an invariant measure. So if there is a unique invariant measure, then there is a unique closed invariant subset M and the complement has measure zero. If the invariant measure is full, then $M = X$. If x is an atom for an invariant measure μ, then the orbit of x consists of atoms with equal measure. Since the measure is finite, x is a periodic point whose orbit must be all of X since f is minimal. That is, X is finite or μ is nonatomic.

(b) Haar measure is the unique measure preserved by all translations and so it is h_g invariant. On the other hand, if, for any measure μ on Γ, we define $Inv_\mu \subset \Gamma$ to be those elements whose translation maps preserve μ, then Inv_μ is a closed subgroup of Γ. If the generator $g \in Inv_\mu$, then $Inv_\mu = \Gamma$ and so μ is the Haar measure μ_Γ.

(c) (4.14) implies that G maps any atom of μ_1 to an atom of μ_2 and that $\mu_2(X_2 \setminus G(X_1)) = 0$. Hence, if μ_2 is nonatomic, then μ_1 is and if μ_2 is full, then $X_2 \setminus G(X_1) = \emptyset$. Since $Inj_G = G^{-1}(G(Inj_G))$ (4.15) follows from (4.14). If the conditions of (4.15) hold, then for any closed subset A of X_1, $G(A)$ is a closed subset of X_2 with $A \cap Inj_G = G^{-1}(G(A) \cap G(Inj_G))$. Hence, (4.15) implies

$$(4.17) \qquad \mu_1(A) \ = \ \mu_1(A \cap Inj_G) \ = \ \mu_2(G(A) \cap G(Inj_G)) \ = \ \mu_2(G(A)).$$

Now if μ_1 is full and A has nonempty interior, then $\mu_1(A) > 0$ implies A meets Inj_G and so Inj_G is dense and G is almost one-to-one. On the other hand, if G is almost one-to-one, then by Lemma 1.1 G is almost open. So if μ_2 is full and A has nonempty interior, then $G(A)$ has nonempty interior and so $\mu_2(G(A)) > 0$ implies $\mu_1(A) > 0$ and so μ_1 is full.

The result when $G(Inj_G)$ has a countable complement is obvious.

(d) If μ_1 is an invariant measure for f_1, then (4.14) implies that $G_*\mu_1$ is an invariant measure for f_2. If μ_2 is the unique invariant measure for f_2, then (4.14) holds. Now assume, in addition, that μ_2 is full and nonatomic and G induces a measure isomorphism. From (c) it follows that G is an almost one-to-one surjection and that μ_1 is full and nonatomic. From (4.17) it is clear that μ_1 is determined by μ_2 and so f_1 is uniquely ergodic. $\qquad \square$

Corollary 4.5. *Let $(X, \leq)$ be an ordered Cantor space and let μ be a measure adapted to $(X, \leq)$. If α is an irrational element of $S^\pi(\mu)$, then the automorphism $h_\alpha \in H_\mu(X)$ is uniquely ergodic, where h_α is the lift under F_μ of the α translation map on the circle.*

Proof. Since α is irrational, $(\mathbb{R}/\mathbb{Z}, \alpha)$ is a compact pointed monothetic group and so the translation map H_α^π is uniquely ergodic by Proposition 4.4(b). Since F_μ^π is injective on the complement of the countable set $F_\mu^{-1}(\tilde{S}(\mu))$, it induces a measure isomorphism from the full nonatomic measure μ on X to the Haar measure, i.e. Lebesgue measure on the circle. By Proposition 4.4(d) the lift h_α is uniquely ergodic. $\qquad \square$

Theorem 4.6. *Let X be a Cantor space. A measure $\mu \in \mathcal{M}_X$ is good on X, i.e. it satisfies the Subset Condition, iff there exists $f \in H_\mu(X)$ which is a uniquely ergodic homeomorphism on X.*

Proof. Assume that μ is good. If $S(\mu) \subset \mathbb{Q}$, then by Theorem 2.16 the measure μ is homeomorphic to the Haar measure on the adding machine $(\Gamma_D, 1)$ with $D = Rec(S(\mu))$. By Proposition 4.4(b) the translation by 1 on Γ is uniquely ergodic. On the other hand, if there exists an irrational in $S(\mu)$, then let α be the corresponding irrational element of $S^\pi(\mu)$. By Theorem 2.7 we can choose an order on X adapted to μ. By Corollary 4.5 the translation map h_α on X is uniquely ergodic.

The converse result is due to Glasner and Weiss. Lemma 2.5 of Glasner and Weiss [1995] implies that if f is a uniquely ergodic homeomorphism on a Cantor space X, then the invariant measure μ is good on X. $\qquad\square$

We have seen that if μ is a good measure on a Cantor space X and $\alpha \in S^\pi(\mu)$, then there exists $f \in H_\mu(X)$ which has the α translation map of the circle as a factor. We do not know if this characterizes the set $S(\mu)$. That is, do there exist μ invariant maps on X which factor over an α translation for any α not in $S^\pi(\mu)$?

It is possible to extend Theorem 2.16 to get a factorization result over adding machines.

Theorem 4.7. *Assume that μ is a good measure on a Cantor space X such that $D = Rec(S(\mu))$ is infinite. Let $(\Gamma_D, 1)$ be the adding machine associated with D with h the 1 translation map on Γ_D. There exists $G : X \to \Gamma_D$ with $X \setminus Inj_G$ countable and $f \in H_\mu(X)$ such that $G \circ f = h \circ G$. The homeomorphism f on X is uniquely ergodic.*

Proof. We will sketch the proof leaving the details to the reader.

Choose an order on X adapted to μ. Let $F_\mu : X \to I$ be the CDF. We will use the inverse limit construction for Γ_D. To do so, let $\{m_k : k = 0, 1, ...\}$ be a strictly increasing sequence of positive integers, cofinal in D, with $m_0 = 1$ and $m_{k-1}|m_k$ for $k = 1, 2, ...$ For each $k = 1, 2, ...$ let d_k be the positive integer m_k/m_{k-1} so that $d_k > 1$.

We will use the phrase "the n pieces of the interval $[a, b]$" to mean the intervals $\{[a + id, a + (i + 1)d] : i = 0, 1, ..., n - 1\}$ with $d = (b - a)/n$ listed in ascending order.

Let $\{A_{1,1}, ..., A_{1,d_1-1}\}$ list the first $d_1 - 1$ of the d_1 pieces of I excluding the last interval A_1. Similarly, let $\{B_{1,1}, ..., B_{1,d_1-1}\}$ list the last $d_1 - 1$ of the d_1 pieces of I excluding the first interval B_1. The excluded intervals both have length $1/m_1$.

Now let $\{A_{2,1}, ..., A_{2,d_2-1}\}$ list the first $d_2 - 1$ of the d_2 pieces of A_1 excluding the last interval A_2. Similarly, let $\{B_{2,1}, ..., B_{2,d_1-1}\}$ list the last $d_2 - 1$ of the d_2 pieces of B_1 excluding the first interval B_2. The excluded intervals both have length $1/m_2$.

Continuing this inductive labeling we obtain two sequences of intervals $\{A_{k,j}\}$, $\{B_{k,j}\}$, converging to the points 1 and 0, respectively, with $|A_{k,j}| = |B_{k,j}| = 1/m_k$. Since $D = Rec(S(\mu))$ and μ is adapted to the order, the reciprocal of each m_k lies in $\tilde{S}(\mu)$ and each of these is a μ interval. Because the intervals in each sequence are nonoverlapping, the lifted sequences $\{\hat{A}_{k,j}\}$ and $\{\hat{B}_{k,j}\}$ form a countable partition of $X \setminus \{M\}$ and $X \setminus \{m\}$, respectively. Use Lemma 4.2 to define f on $\hat{A}_{k,j}$ as the unique order-preserving map to $\hat{B}_{k,j}$ and finally let $f(M) = m$. Clearly, $f \in H_\mu(X)$

and it is easy to check that it is a cyclic permutation of the lifts of the m_k pieces of I. This provides the projection G from X to the inverse limit Γ_D which maps f to the translation by 1.

Finally, the intervals provide an ordering on Γ_D and it is easy to see that G is the unique continuous map such that

$$(4.18) \qquad\qquad F_\mu \;=\; F_{\mu_\Gamma} \circ G.$$

Hence, $Inj_{F_\mu} \subset Inj_G$ and so $X \setminus Inj_G$ is countable. It follows from Proposition 4.4(d) that f is uniquely ergodic. $\qquad\square$

Remark. Let (Γ, g) be a pointed Cantor group. By Proposition 1.5 and (2.50) in Theorem 2.16 the measure μ can only be mapped to Haar measure μ_Γ when $S(\mu_\Gamma) \subset S(\mu) \cap \mathbb{Q}$ and so by Theorem 2.17 any such pointed Cantor group is a homomorphic image of $(\Gamma_D, 1)$ with $D = Rec(S(\mu))$.

Theorem 4.8. *Let μ be a good measure on a Cantor space X such that $S(\mu)$ is field-like. Assume that $I \subset S(\mu)$ is such that $I \cup \{1\}$ is linearly independent over $\mathbb{Q}$ and let $\pi(I)$ be the corresponding subset of $S^\pi(\mu) \subset \mathbb{R}/\mathbb{Z}$. Let $H = \prod_{\alpha \in \pi(I)} H_\alpha^\pi$ be the translation map on the torus $\mathbf{T}_I$ (the product of copies of the circle $\mathbb{R}/\mathbb{Z}$ indexed by $\pi(I)$).*

There exists $h \in H_\mu(X)$ and an almost one-to-one measure isomorphism $G : X \to \mathbf{T}_I$ such that $G \circ h = H \circ G$. G maps μ to the Haar measure on $\mathbf{T}_I$ and the homeomorphism h on X is uniquely ergodic.

Proof. Let $g \in \mathbf{T}_I$ with $g_\alpha = \alpha$. The rational independence of $I \cup \{1\}$ implies that $(\mathbf{T}_I, g)$ is a compact pointed monothetic group and that H is the g translation map. By Proposition 4.4(b) H is uniquely ergodic.

Let Y be the product of copies of X indexed by $\pi(I)$ with ν the measure on Y which is the product of copies of μ. Let $\tilde{G} : Y \to \mathbf{T}_I$ be the product of copies of $F_\mu^\pi : X \to \mathbb{R}/\mathbb{Z}$. Inj_G is the product of the dense sets Inj_{F_μ} and so it is dense, i.e. $\tilde{G}$ is an almost one-to-one surjection. The complement is the union of sets indexed by $\pi(I)$, where the points of the α set have α coordinate in $X \setminus Inj_{F_\mu}$. Hence, $Y \setminus Inj_{\tilde{G}}$ has measure zero, i.e. $\tilde{G}$ is a measure isomorphism. Let $\tilde{h} \in H_\nu(Y)$ be the product of copies of h_α from Theorem 4.3 so that $\tilde{G} \circ \tilde{h} = H \circ \tilde{G}$. By Proposition 4.4(d) $\tilde{h}$ is uniquely ergodic.

Because $S(\mu)$ is field-like, Corollary 3.7 implies there exists a homeomorphism $q : X \to Y$ which maps μ to ν. We let $h = q^{-1} \circ \tilde{h} \circ q$ and $G = \tilde{G} \circ q$. $\qquad\square$

Theorem 4.9. *Let μ be a good measure on a Cantor space X such that $S(\mu)$ is field-like. Assume that $\alpha_1, ..., \alpha_n \in S(\mu) \setminus \{0, 1\}$ with $\sum_k \alpha_k = 1$. Let $[n] =_{def} \{1, ..., n\}$ and let ν_n be the measure on $[n]$ with $\nu_n(\{k\}) = \alpha_k$ for $k = 1, ..., n$. Let $Y = [n]^\mathbb{Z}$ with the product topology and let ν be the product measure on Y with copies of ν_n on each factor. Let H be the shift homeomorphism on Y, i.e. $H(y)_i = y_{i+1}$ for all $i \in \mathbb{Z}$. Thus, $\nu \in \mathcal{M}_X$ and $H \in H_\nu(Y)$.*

There exists $h \in H_\mu(X)$ and an almost one-to-one measure isomorphism $G : X \to Y$ such that G maps μ to ν and $G \circ h = H \circ G$.

Proof. Choose an order on X adapted to μ. Let $F_\mu : X \to I$ be the CDF. For $k = 1, ..., n$ let J_k denote the the μ interval $[\sigma_{k-1}, \sigma_k]$ with $\sigma_0 = 0$ and $\sigma_k =_{def} \sum_{j \le k} \alpha_j$. Thus, the F_μ-lifts $\hat{J}_k$ for $k = 1, ..., n$ forms a partition of X with $\mu(\hat{J}_k) = \alpha_k$ for

all k. Because $S(\mu)$ is field-like and μ is adapted to the order, $\tilde{S}(\mu_{J_k}) = S(\mu_{J_k}) = S(\mu) = \tilde{S}(\mu)$ and so there exists a unique order isomorphism $h_k : (\hat{J}_k, \leq) \to (X, \leq)$ which maps $\mu_{\hat{J}_k}$ to μ. In the product space $X \times X$ the homeomorphism

$$(4.19) \qquad h_k^\times \quad =_{def} \quad h_k \times (h_k)^{-1} : \hat{J}_k \times X \to X \times \hat{J}_k$$

maps $\mu_{\hat{J}_k} \times \mu$ to $\mu \times \mu_{\hat{J}_k}$. Since the $\mu \times \mu$ measure of each of these pieces is α_k we see that $h_k^\times$ is $\mu \times \mu$ measure preserving on each slice. Putting together these pieces we obtain a map $\tilde{h} \in H_{\mu \times \mu}(X \times X)$ which maps the vertical partition to the horizontal partition. Define $\tilde{G} : X \times X \to Y$ by

$$(4.20) \qquad \tilde{G}(x, y)_i \;=\; k \qquad \Longleftrightarrow \qquad (\tilde{h})^i(x, y) \in \hat{J}_k \times X \quad \text{for } i \in \mathbb{Z}.$$

It is easy to see that $\tilde{G}$ maps $\tilde{h}$ to H and $\mu \times \mu$ to ν. Furthermore, if $\tilde{G}(x_1, y_1) = \tilde{G}(x_2, y_2)$, then $F_\mu(x_1) = F_\mu(x_2)$ and $F_\mu(y_1) = F_\mu(y_2)$ because when $X \times X$ is projected to $I \times I$ by $F_\mu \times F_\mu$, the map $\tilde{h}$ becomes the usual Baker's Transformation associated with $\alpha_1, ..., \alpha_n$. Hence, $\tilde{G}$ is an almost one-to-one surjection and a measure isomorphism from $\mu \times \mu$ to ν.

Because $S(\mu)$ is field-like, Theorem 3.6 implies there exists a homeomorphism $q : X \to X \times X$ which maps μ to $\mu \times \mu$. We let $h = q^{-1} \circ \tilde{h} \circ q$ and $G = \tilde{G} \circ q$. $\qquad \square$

If on a Cantor space X we are given a measure $\mu \in \mathcal{M}_X$, a nonempty clopen subset V of X and a positive integer n, then a partition $\mathcal{A}$ of V is called a $1/n$ *partition of V* when

$$(4.21) \qquad A \in \mathcal{A} \qquad \Longrightarrow \qquad \mu(A) \;=\; \mu(V)/n.$$

Clearly a $1/n$ partition has cardinality n.

Recall that (Ξ, g) denotes the *universal adding machine*, i.e. the Cantor group with D_Ξ the entire set of positive integers, i.e. the Cantor group with clopen subgroups of every positive index. By (2.50) and Theorem 2.16, the Haar measure μ_Ξ is characterized up to homeomorphism as the unique good measure with clopen values set

$$(4.22) \qquad S(\mu_\Xi) \quad = \quad \mathbb{Q} \cap I.$$

Lemma 4.10. *Let μ be a good measure on a Cantor space X.*

  (a) $\mathbb{Q} \cap I \subset S(\mu)$ *iff X admits a $1/n$ partition for every positive integer n.*
  (b) *The following conditions are equivalent:*
    (1) $S(\mu)$ *is $\mathbb{Q}$-like.*
    (2) *For every nonempty clopen subset V of X $S(\mu_V)$ is $\mathbb{Q}$-like.*
    (3) *For every nonempty clopen subset V of X $\mathbb{Q} \cap I \subset S(\mu_V)$.*
    (4) *Every nonempty clopen subset V of X admits a $1/n$ partition for every positive integer n.*
    (5) *With $\mu_\Xi \times \mu$ the product measure on $\Xi \times X$, $S(\mu_\Xi \times \mu) = S(\mu)$.*
    (6) *There exists a homeomorphism $q : X \to \Xi \times X$ such that $q_* \mu = \mu_\Xi \times \mu$.*

Proof. (a) If X admits a $1/n$ partition, then $n \in Rec(S(\mu))$. Conversely, if $n \in R(S(\mu))$, then since $S(\mu)$ is group-like, $k/n \in S(\mu)$ for $k = 0, 1, ..., n$. Choose an order on X adapted to μ. As in the proof of Theorem 4.9, $J_k = [(k-1)/n, k/n]$ is a μ interval for $k = 1, ..., n$ and $\{\hat{J}_k : k = 1, ..., n\}$ is a $1/n$ partition of X. Hence, X admits such partitions for all n iff $\mathbb{Z}^* = Rec(S(\mu))$ and so by (2.12) iff $\mathbb{Q} \cap I \subset S(\mu)$.

(b) $(1) \Rightarrow (6)$: Theorem 3.6.

$(6) \Rightarrow (5)$: Proposition 1.5.

$(5) \Rightarrow (1)$: Let $\alpha \in \mathbb{Q}^* \cap I$ and $\beta \in S(\mu)$. There exist clopens $U \subset \Xi$ and $V \subset X$ such that $\mu_\Xi(U) = \alpha$ and $\mu(V) = \beta$. By assumption (5) $\alpha\beta = \mu_\Xi \times \mu(U \times V) \in S(\mu)$. It follows that the group $S(\mu) + \mathbb{Z}$ is closed under multiplication by elements of $\mathbb{Q}$, i.e. $S(\mu)$ is $\mathbb{Q}$-like.

$(1) \Rightarrow (2)$: Apply (3.3) and Lemma 2.1(d).

$(2) \Rightarrow (3)$: Obvious.

$(3) \Rightarrow (1)$: Let $\alpha \in \mathbb{Q}^* \cap I$ and $\beta = \mu(V)$ with V clopen in X. By assumption (3), $\alpha \in S(\mu_V)$ and so $\alpha\beta \in S(\mu)$.

$(3) \Leftrightarrow (4)$: Apply part (a) to each V. $\square$

For a partition $\mathcal{A}$ on a space X the associated equivalence relation $\equiv_\mathcal{A}$ is defined to be $\bigcup\{A \times A : A \in \mathcal{A}\}$. If $f_1, f_2 : Y \to X$ are functions, then we write $f_1 \equiv_\mathcal{A} f_2$ if $f_1(y) \equiv_\mathcal{A} f_2(y)$ for all $y \in Y$, i.e. for every $y \in Y$ there exists $A \in \mathcal{A}$ such that $\{f_1(y), f_2(y)\} \subset A$. If $\mathcal{A}$ satisfies $mesh(\mathcal{A}) \leq \epsilon$, then $f_1 \equiv_\mathcal{A} f_2$ implies $d(f_1(y), f_2(y)) \leq \epsilon$ for all $y \in Y$, i.e. f_1 and f_2 are uniformly ϵ close. On the other hand, if $\epsilon > 0$ is the minimum distance between two points in different elements of $\mathcal{A}$, then $d(f_1(y), f_2(y)) < \epsilon$ for all $y \in Y$ implies $f_1 \equiv_\mathcal{A} f_2$.

If on a Cantor space X we are given a measure $\mu \in \mathcal{M}_X$, a nonempty clopen subset V of X and a numbered partition $\{A_1, ..., A_n\}$ of V, then $h \in H_\mu(X)$ is said to be *cyclic on V of period n with respect to $\{A_1, ..., A_n\}$* if

$$(4.23) \qquad \begin{aligned} h^n(x) &= x \qquad \text{for all} \quad x \in V \\ \text{and} \qquad h(A_i) &= A_{i+1} \end{aligned}$$

for $i = 1, ..., n$ with addition mod n. Since h is measure preserving, $\{A_1, ..., A_n\}$ must be a $1/n$ partition of V.

A homeomorphism $h \in H_\mu(X)$ is said to be *cyclic on V of period n* if such a partition exists. It is called a *cyclic map of period n* if it is a cyclic map of period n on $V = X$ and it is called a *cyclic map* if it is a cyclic map of period n for some, necessarily unique, positive integer n.

If μ is good on X and $\{A_1, ..., A_n\}$ is an arbitrary numbered $1/n$ partition of V, then there exist for $i = 1, ..., n - 1$ homeomorphisms $h_i : A_i \to A_{i+1}$ mapping μ_{A_i} to $\mu_{A_{i+1}}$. Define

$$(4.24) \qquad\qquad h_n = (h_{n-1} \circ ... \circ h_1)^{-1} : A_n \to A_1$$

and extend by any measure preserving automorphism of $X \setminus V$, e.g. $1_{X \setminus V}$, to get $h \in H_\mu(X)$ which is cyclic on V with respect to $\{A_1, ..., A_n\}$.

Lemma 4.11. *Let X be Cantor space and $\mu \in \mathcal{M}_X$. Assume that $h \in H_\mu(X)$, V is a nonempty clopen subset of X and $\{A_1, ..., A_n\}$ is a numbered partition of V such that $h(A_i) \subset A_{i+1}$ for $i = 1, ..., n$ with addition mod n.*

(a) *$h(A_i) = A_{i+1}$ for $i = 1, ..., n$ with addition mod n. If for all $x \in A_n, h^n(x) = x$, then h is cyclic on V of period n with respect to $\{A_1, ..., A_n\}$.*

(b) *Assume that μ is a good measure on X with $S(\mu)$ $\mathbb{Q}$-like and that h is cyclic on V of period n with respect to $\{A_1, ..., A_n\}$. For any positive integer k and any partition $\mathcal{B}$ on X, there exists a numbered partition $\{\tilde{A}_1, ...\tilde{A}_{kn}\}$ of V and $\tilde{h} \in H_\mu(X)$ such that*

(1) *For $j = 1, ..., kn$ and $i = 1, ..., n$, if $i \equiv j \mod n$, then $\tilde{A}_j \subset A_i$.*

(2) $\tilde{h}(\tilde{A}_j) \subset \tilde{A}_{j+1}$ for $j = 1, ..., kn$ with addition mod kn.
(3) $\tilde{h}^{kn}(x) = x$ for all $x \in V$.
(4) $\tilde{h}(x) = h(x)$ for all $x \in X \setminus A_n$.
(5) $\tilde{h}(x) \equiv_{\mathcal{B}} h(x)$ for all $x \in A_n$.
In particular, $\tilde{h}$ is cyclic on V of period kn with respect to the refinement $\{\tilde{A}_1, ..., \tilde{A}_{kn}\}$ of $\{A_1, ..., A_n\}$ and $\tilde{h} \equiv_{\mathcal{B}} h$.

Proof. (a) Since $h \in H_\mu$, $\mu(A_i) = \mu(h(A_i)) \le \mu(A_{i+1})$ for $i = 1, ..., n$ with addition mod n. Since these inequalities cycle they are equations. Hence the clopen sets $A_{i+1} \setminus h(A_i)$ have measure zero for $i = 1, ..., n-1$. Since μ is full these sets are empty.

If $h^n = 1_X$ on A_n and $y \in V$, then $y \in A_{n-i}$ for some $i = 0, ..., n$ and so $x = h^i(y) \in A_n$. Hence, $h^{n+i}(y) = h^n(x) = x = h^i(y)$. Applying h^{-i} we see that $h^n(y) = y$.

(b) Because $S(\mu)$ $\mathbb{Q}$-like, on every nonempty clopen of the form $B \cap A_n$ with $B \in \mathcal{B}$ we can choose a measure preserving map which is cyclic of period k on $B \cap A_n$. Assemble these to obtain a measure preserving map g on A_n and extend g by the identity on $X \setminus A_n$. Thus, $g(x) = x$ for $x \in X \setminus A_n$ and $g(x) \equiv_{\mathcal{B}} x$ for all $x \in A_n$. On A_n g is cyclic of period k with respect to some numbered partition $C_1, ..., C_k$ of A_n. Let $\tilde{h} = g \circ h$. Clearly, (4) and (5) hold. For $x \in A_n$ we have

$$(4.25) \qquad \tilde{h}^i(x) = h^i(x) \quad \text{for } 0 \le i < n \qquad \text{and} \quad \tilde{h}^n(x) = g(x).$$

Define the numbered partition on V by

$$(4.26) \qquad \tilde{A}_{i+(r-1)n} =_{def} h^{i-n}(C_r) \qquad \text{for } i = 1, ..., n \text{ and } r = 1, ..., k.$$

Obviously (1) holds, and from (4.25), (2) and (3) hold as well. $\qquad \square$

Now we consider a measure μ on a Cantor space X with $S(\mu)$ $\mathbb{Q}$-like.

Given any partition $\mathcal{A}$ on X the measure μ induces the positive distribution $\mu_{\mathcal{A}}$ on $\mathcal{A}$ by

$$(4.27) \qquad\qquad \mu_{\mathcal{A}}(A) = \mu(A) \qquad \text{for } A \in \mathcal{A}.$$

If $f \in H_\mu(X)$, then f and μ induce a distribution $P_{\mathcal{A}}[f]$ on the product set $\mathcal{A} \times \mathcal{A}$ by

$$(4.28) \quad P_{\mathcal{A}}[f](A_1, A_2) = \mu(A_1 \cap f^{-1}(A_2)) = \mu(f(A_1) \cap A_2) \qquad \text{for } A_1, A_2 \in \mathcal{A}.$$

Clearly,

$$(4.29) \qquad \sum_{B \in \mathcal{A}} P_{\mathcal{A}}[f](A, B) = \sum_{B \in \mathcal{A}} P_{\mathcal{A}}[f](B, A) = \mu(A) \qquad \text{for } A \in \mathcal{A}.$$

The associated relation $R_{\mathcal{A}}[f]$ on $\mathcal{A}$ is the subset of $\mathcal{A} \times \mathcal{A}$ given by
$$(4.30)$$
$$R_{\mathcal{A}}[f] = \{(A_1, A_2) : A_1 \cap f^{-1}(A_2) \ne \emptyset\} = \{(A_1, A_2) : P_{\mathcal{A}}[f](A_1, A_2) > 0\}.$$

Lemma 4.12. *The transitive relation $\mathcal{O}R_{\mathcal{A}}[f] =_{def} \bigcup_{k=1}^\infty (R_{\mathcal{A}}[f])^k$ is an equivalence relation.*

Proof. From (4.29) it follows that $R = R_{\mathcal{A}}[f]$ is a surjective relation. That is, for every $A \in \mathcal{A}$ there exist B_1, B_2 such that $P_{\mathcal{A}}[f](A, B_1)$ and $P_{\mathcal{A}}[f](B_2, A)$ are positive. We prove that $\mathcal{O}R$ is an equivalence relation on $\mathcal{A}$ by induction on the cardinality n of $\mathcal{A}$.

If $n = 1$, then $\mathcal{A} = \{X\}$ and $R = \{(X, X)\}$ is the trivial equivalence relation.

On the *cyclic set* of $\mathcal{O}R$,

(4.31) $|\mathcal{O}R| \quad =_{def} \quad \{A \in \mathcal{A} : (A, A) \in \mathcal{O}R\},$

$\mathcal{O}R \cap (\mathcal{O}R)^{-1}$ is an equivalence relation and its equivalence classes, the R basic sets, are partially ordered by $\mathcal{O}R$. Choose $\mathcal{B}$ a terminal basic set. That is, $\mathcal{B} \times \mathcal{B} \subset \mathcal{O}R$ and

(4.32) $B \in \mathcal{B} \quad \text{and} \quad (B, A) \in \mathcal{O}R \quad \implies \quad A \in \mathcal{B}.$

From (4.29) and (4.32) we have

(4.33)
$$\sum_{B \in \mathcal{B}} \sum_{A \in \mathcal{A}} P_{\mathcal{A}}[f](A, B) \;=\; \sum_{B \in \mathcal{B}} \mu(B)$$
$$= \sum_{B \in \mathcal{B}} \sum_{A \in \mathcal{A}} P_{\mathcal{A}}[f](B, A) \;=\; \sum_{A, B \in \mathcal{B}} P_{\mathcal{A}}[f](B, A),$$

since $B \in \mathcal{B}$ and $A \notin \mathcal{B}$ implies $P_{\mathcal{A}}[f](B, A) = 0$. It follows that $P_{\mathcal{A}}[f](A, B) = 0$ if $B \in \mathcal{B}$ and $A \notin \mathcal{B}$.

Now if $\mathcal{A} = \mathcal{B}$, then $\mathcal{O}R = \mathcal{B} \times \mathcal{B}$ is an equivalence relation. If $\tilde{\mathcal{A}} = \mathcal{A} \setminus \mathcal{B}$ is nonempty, then it is a partition of cardinality less than n on a nonempty clopen subset V of X. Furthermore, $P_{\mathcal{A}}[f](B, A) = P_{\mathcal{A}}[f](A, B) = 0$ if $A \in \tilde{\mathcal{A}}$ and $B \in \mathcal{B}$. Hence, $f(V) = V$ and $f \in H_{\mu_V}(V)$. Applying the inductive hypothesis we have that $\mathcal{O}R_{\tilde{\mathcal{A}}}[f]$ is an equivalence relation and so

(4.34) $\mathcal{O}R_{\mathcal{A}}[f] \quad = \quad \mathcal{O}R_{\tilde{\mathcal{A}}}[f] \;\cup\; (\mathcal{B} \times \mathcal{B})$

is an equivalence relation as well. $\square$

Theorem 4.13. *If μ is a good measure on a Cantor space X such that $S(\mu)$ is $\mathbb{Q}$-like, then the set of cyclic functions is dense in $H_{\mu}(X)$.*

Proof. Given a partition $\mathcal{A}$, let P denote the $\mathcal{A} \times \mathcal{A}$ matrix $P_{\mathcal{A}}[f]$. We use induction on number N of nonzero entries in P to prove that there exists a cyclic function g in H_{μ} such that $g \equiv_{\mathcal{A}} f$.

If $N = 1$, then $\mathcal{A} = \{X\}$ and $g = 1_X$ is a cyclic function with $g \equiv_{\mathcal{A}} f$.

For the inductive step we first construct a cycle on a nonempty clopen part of the space.

Case i: Some diagonal entry $P(A_1, A_1)$ is positive.

Define $Z_1 = A_1 \cap f^{-1}(A_1)$. By (4.28) $\mu(Z_1) = P(A_1, A_1)$. Since $f(Z_1) = f(A_1) \cap A_1$ and $f \in H_{\mu}$ we have $\mu(A_1 \setminus f(Z_1)) = \mu(A_1 \setminus Z_1)$. Because μ is good we can choose $h \in H_{\mu}$ such that

(4.35)
$$h(A_1 \setminus f(Z_1)) \quad = \quad A_1 \setminus Z_1,$$
$$h \;=\; f^{-1} \quad \text{on} \quad f(Z_1),$$
$$h \;=\; 1_X \quad \text{on} \quad X \setminus A_1.$$

Clearly, $h \equiv_{\mathcal{A}} 1_X$ and so with $f_1 =_{def} h \circ f$ $f_1 \equiv_{\mathcal{A}} f$. Let $Z = Z_1$. On Z f_1 restricts to the identity. That is, it is cyclic of period $n = 1$.

Case ii: With $A_1 \neq A_2 \in \mathcal{A}$, $p =_{def} P(A_1, A_2)$ is the smallest positive entry of P.

By Lemma 4.12 we can construct a chain $\{A_1, A_2, ..., A_n\}$ in $\mathcal{A}$ so that with $A_{n+1} =_{def} A_1$ we have $P(A_k, A_{k+1}) > 0$ for $k = 1, 2, ..., n$. By assumption, $P(A_k, A_{k+1}) \geq p$ for $k = 1, 2, ..., n$. Suppose $n \geq 2$ is the minimal length of a chain

satisfying these properties. If for some $2 \leq j < k \leq n$, $A_j = A_k$, then we could re-move $A_{j+1}, ..., A_k$ to get a shorter list. Similarly, if for some $2 < k \leq n$, $A_k = A_1$ we could remove $A_k, ..., A_n$. Thus, the chain of minimal length has distinct members in $\mathcal{A}$.

Define $Z_1 = A_1 \cap f^{-1}(A_2)$ so that $f(Z_1) \subset A_2$ and

$$(4.36) \qquad \mu(f(Z_1)) \quad = \quad p \quad \leq \quad \mu(A_2 \cap f^{-1}(A_3)).$$

By Proposition 2.11 applied to the good measure μ_{A_2} there exists $h_2 \in H_{\mu_{A_2}}(A_2)$ such that

$$(4.37) \qquad Z_2 \quad =_{def} \quad h_2(f(Z_1)) \quad \subset \quad A_2 \cap f^{-1}(A_3),$$

and so $\mu(Z_2) = p$.

Inductively, for $k = 3, ..., n$ we construct $h_k \in H_{\mu_{A_k}}(A_k)$ and $Z_k \subset A_k \cap f^{-1}(A_{k+1})$ with $\mu(Z_k) = p$ such that

$$(4.38) \qquad Z_k \quad =_{def} \quad h_k(f(Z_{k-1})) \quad \subset \quad A_k \cap f^{-1}(A_{k+1}).$$

Since $A_{n+1} = A_1$ (4.38) implies that Z_1 and $f(Z_n)$ are clopen subsets of A_1 with the same measure. We can choose $h \in H_\mu$ such that

$$h(A_1 \setminus f(Z_n)) \quad = \quad A_1 \setminus Z_1,$$

$$h \quad = \quad (f \circ h_n \circ f \circ h_{n-1} \circ ... \circ h_2 \circ f)^{-1} \quad \text{on} \quad f(Z_n),$$

$$(4.39) \qquad h \quad = \quad h_k \quad \text{on} \quad A_k \quad \text{for } k = 2, ..., n,$$

$$h \quad = \quad 1_X \quad \text{on} \quad X \setminus (\bigcup_{j=1}^{n} A_j).$$

Clearly, $h \equiv_{\mathcal{A}} 1_X$ and so with $f_1 =_{def} h \circ f$, $f_1 \equiv_{\mathcal{A}} f$. With respect to the numbered partition $\{Z_1, ..., Z_n\}$ of $Z =_{def} \bigcup_{j=1}^{n} Z_j$, f_1 is cyclic on Z of period n.

So in each case, we have constructed $f_1 \equiv_{\mathcal{A}} f$ cyclic of period n on the clopen subset Z of X.

If $Z = X$ we have constructed the desired cyclic function on X. It remains to consider the case when $\tilde{X} = X \setminus Z$ is a nonempty clopen subset with the good measure $\tilde{\mu} = \mu_{\tilde{X}}$ and the partition $\tilde{\mathcal{A}} = \{A \cap \tilde{X} : A \in \mathcal{A}\} \setminus \{\emptyset\}$. Notice that Z is an invariant subset for f_1 and so letting $\tilde{f}$ denote the restriction of f_1 to $\tilde{X}$ we have $\tilde{f} \in H_{\tilde{\mu}}(\tilde{X})$. So if we let

$$(4.40) \qquad \tilde{P}(A, B) \quad =_{def} \quad \mu(A \cap f_1^{-1}(B) \cap \tilde{X})/\mu(\tilde{X}),$$

then $\tilde{P}$ is the matrix $P_{\tilde{\mathcal{A}}}[\tilde{f}]$ extended by 0 for those indices of $A \in \mathcal{A}$ such that $A \cap \tilde{X} = \emptyset$. Since $f_1 \equiv_{\mathcal{A}} f$ we have for all $A, B \in \mathcal{A}$

$$(4.41) \qquad A \cap f^{-1}(B) \quad = \quad A \cap f_1^{-1}(B).$$

Consequently,

$$(4.42) \qquad \tilde{P}(A, B) \quad \leq \quad P(A, B)/\mu(\tilde{X}).$$

Furthermore, since Z_1 is disjoint from $\tilde{X}$,

$$(4.43) \qquad \begin{aligned} \tilde{P}(A_1, A_1) &= 0 \quad \text{in Case i,} \\ \tilde{P}(A_1, A_2) &= 0 \quad \text{in Case ii.} \end{aligned}$$

Thus, $P_{\tilde{\mathcal{A}}}[\tilde{f}]$ has fewer positive entries than does $P_{\mathcal{A}}[f]$. It follows from the induction hypothesis that there exists $\tilde{f}_1 \in H_{\tilde{\mu}}(\tilde{X})$ such that $\tilde{f}_1 \equiv_{\tilde{\mathcal{A}}} \tilde{f}$ and $\tilde{f}_1$ is

cyclic on $\tilde{X}$ with period $\tilde{n}$. Define $f_2 \in H_\mu(X)$ to be f_1 on Z and $\tilde{f}_1$ on the complement $\tilde{X} = X \setminus Z$. Thus, $f_2 \equiv_{\mathcal{A}} f$ and f_2 is cyclic of period n on Z and of period $\tilde{n}$ on $\tilde{X}$. Apply Lemma 4.11(a) to change f_2 on each of the two pieces Z and $\tilde{X}$ of X to get $f_3 \equiv_{\mathcal{A}} f_2 \equiv_{\mathcal{A}} f$ which is cyclic of period $n \cdot \tilde{n}$ on each piece. $\square$

Recall that a partition $\mathcal{A}_1$ *refines* $\mathcal{A}_2$, written $\mathcal{A}_1 \to \mathcal{A}_2$, if each element of $\mathcal{A}_1$ is contained in a member of $\mathcal{A}_2$ and we define the surjection $\pi_{\mathcal{A}_1}^{\mathcal{A}_2} : \mathcal{A}_1 \to \mathcal{A}_2$ by

$$(4.44) \qquad\qquad A \subset \pi_{\mathcal{A}_1}^{\mathcal{A}_2}(A) \qquad \text{for} \quad A \in \mathcal{A}_1.$$

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are partitions of X, then the least common refinement is

$$(4.45) \qquad \mathcal{A}_1 \wedge \mathcal{A}_2 \quad =_{def} \quad \{A_1 \cap A_2 : A_1 \in \mathcal{A}_1 \text{ and } A_2 \in \mathcal{A}_2\} \setminus \{\emptyset\}.$$

If f is an automorphism of X, then a partition $\mathcal{A}$ is called *invariant* if

$$(4.46) \qquad\qquad A \in \mathcal{A} \qquad \Longrightarrow \qquad f(A) \in \mathcal{A}.$$

This says exactly that the relation $R_{\mathcal{A}}[f]$ is a function on $\mathcal{A}$ in which case we will denote it by $f_{\mathcal{A}}$. Since the relation $R_{\mathcal{A}}[f]$ is always surjective and $\mathcal{A}$ is finite, $f_{\mathcal{A}}$ is a permutation of the invariant partition $\mathcal{A}$. The partition is called *fixed* if

$$(4.47) \qquad\qquad A \in \mathcal{A} \qquad \Longrightarrow \qquad f(A) = A.$$

That is, $\mathcal{A}$ is invariant and $f_{\mathcal{A}} = 1_{\mathcal{A}}$. At the opposite extreme, the partition is called *cyclic* if it is invariant and the permutation $f_{\mathcal{A}}$ consists of a single cycle on $\mathcal{A}$.

If $\mathcal{A}_1 \to \mathcal{A}_2$ and $\mathcal{A}_1$ is invariant, then $\mathcal{A}_2$ invariant and

$$(4.48) \qquad\qquad \pi_{\mathcal{A}_1}^{\mathcal{A}_2} \circ f_{\mathcal{A}_1} \quad = \quad f_{\mathcal{A}_2} \circ \pi_{\mathcal{A}_1}^{\mathcal{A}_2}.$$

If $\mathcal{A}_1$ is fixed (or cyclic), then $\mathcal{A}_2$ is fixed (resp. cyclic).

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are invariant (or fixed) partitions, then $\mathcal{A}_1 \wedge \mathcal{A}_2$ is invariant (resp. fixed). However, $\mathcal{A}_1 \wedge \mathcal{A}_2$ is not ncessarily cyclic when the two factors are.

If $\mathcal{A}$ is an invariant partition, then we can concatenate the elements of the separate cycles of the permutation $f_{\mathcal{A}}$ to obtain the *fixed partition associated with* $\mathcal{A}$ denoted $[\mathcal{A}]$. Thus, $[\mathcal{A}]$ is the finest fixed partition of which $\mathcal{A}$ is a refinement. $\mathcal{A}$ is a cyclic partition iff it is invariant and $[\mathcal{A}]$ is the trivial partition $\{X\}$.

Definition 4.14. Let X be a Cantor space equipped with a metric d and a measure $\mu \in \mathcal{M}_X$. Let $\mathcal{A} = \{A_1, ..., A_n\}$ be a numbered partition of X, ϵ be a positive rational, N be a positive integer and $\alpha : [n] \to I \setminus \{0, 1\}$ be a function which satisfies

$$(4.49) \qquad\qquad \alpha_i \in S(\mu_{A_i}) \qquad \text{for} \quad i \in [n] = \{1, 2, ..., n\}.$$

$f \in H_\mu(X)$ is in the subset $G(\mathcal{A}, \alpha, \epsilon, N)$ if *either* $\mathcal{A}$ is not a fixed partition for f *or* there exist positive integers k, M and a partition $\mathcal{W}$ indexed by $[n] \times [2] \times [M] \times [kN]$ so that for $ijsr \in [n] \times [2] \times [M] \times [kN]$

   (i) $W_{ijsr} \subset A_i$.
   (ii) $\mu(W_{i1sr}) = \alpha_i \mu(W_{i1sr} \cup W_{i2sr})$.
   (iii) $d(W_{ijsr}) < \epsilon$.
   (iv) $f(W_{ijsr}) = W_{ijs(r+1)}$ \qquad with addition mod kN.

Lemma 4.15. *Each $G(\mathcal{A}, \alpha, \epsilon, N)$ is an open subset of $H_\mu(X)$. If μ is a good measure on X with $S(\mu)$ $\mathbb{Q}$-like, then each $G(\mathcal{A}, \alpha, \epsilon, N)$ is dense in $H_\mu(X)$.*

Proof. If $f \equiv_{\mathcal{A}} f_1$, then the relations $R_{\mathcal{A}}[f]$ and $R_{\mathcal{A}}[f_1]$ agree. So $\mathcal{A}$ is invariant, fixed or cyclic for f_1 if it satisfies the corresponding property for f. Thus, if $\mathcal{A}$ is not fixed for f, then it is not fixed for f_1. On the other hand, for the partition $\mathcal{W}$ conditions (i), (ii) and (iii) do not depend on f while if (iv) holds for f, then it holds for any f_1 such that $f_1 \equiv_{\mathcal{W}} f$. Thus, $G = G(\mathcal{A}, \alpha, \epsilon, N)$ is an open subset of H_μ.

Now assume that μ is good on X with $S(\mu)$ $\mathbb{Q}$-like, $\mathcal{B}$ is an arbitrary partition of X and that $f \in H_\mu(X)$. We will construct $g \equiv_{\mathcal{B}} f$ with $g \in G$. If $\mathcal{A}$ is not fixed for f, then $f \in G$ and we can let $g = f$. Assume now that $\mathcal{A}$ is fixed partition for f. Furthermore, by replacing $\mathcal{B}$ by $\mathcal{B} \wedge \mathcal{A}$ if necessary, we can assume that

$$(4.50) \qquad\qquad\qquad\qquad \mathcal{B} \rightarrow \mathcal{A}.$$

By Theorem 4.13 and Lemma 4.11(b) there exists a positive integer k and a map $g \in H_\mu$ with $g \equiv_{\mathcal{B}} f$ such that g is a cyclic map of period kN. That is, there is a partition $\{W_r : r \in [kN]\}$ with $W_{kN+1} = W_1$ such that $g(W_r) = W_{r+1}$ for all r and $g^{kN} = 1_X$.

Since $\mathcal{B}$ refines $\mathcal{A}$, $g \equiv_{\mathcal{A}} f$ and so $\mathcal{A}$ is a fixed partition for g. So if $W_{ir} = A_i \cap W_r$, then $g(W_{ir}) = W_{i(r+1)}$. Hence, the g orbit of any $x \in A_i$ moves cyclically through the W_{ir}'s and so no W_{ir} is empty.

Choose $\delta > 0$ an ϵ modulus of uniform continuity for $1_X, g, ..., g^{kN}$. Because $S(\mu)$ is $\mathbb{Q}$-like, we can choose a positive integer M large enough that each W_{i1} can be partitioned into exactly M clopen pieces of equal measure and of diameter less than δ. These pieces are labeled W_{is1} for $s \in [M]$. Define $W_{isr} = g^{r-1}(W_{is1})$ for $r \in [kN]$. Thus,

$$(4.51) \qquad\qquad d(W_{isr}) < \epsilon \qquad \text{and} \qquad \mu(W_{isr}) = \mu(A_i)/kNM.$$

By assumption on the map α, each $\mu(A_i)\alpha_i \in S(\mu)$. Since $S(\mu)$ is $\mathbb{Q}$-like, $\mu(A_i)\alpha_i/kNM \in S(\mu)$. So by the Subset Condition we can choose a clopen $W_{i1s1} \subset W_{is1}$ with this measure and let $W_{i2s1} = W_{is1} \setminus W_{i1s1}$. Since $\alpha_i \neq 0, 1$ these sets are nonempty. Finally, define $W_{ijsr} = g^{r-1}(W_{ijs1})$ for $r \in [kN]$ so that

$$(4.52) \qquad\qquad\qquad \mu(W_{i1sr}) = \alpha_i \mu(W_{isr}).$$

Notice that $g^{kN} = 1_X$ implies that $g(W_{ijs(kN)}) = W_{ijs1}$ as required. $\qquad\square$

Define the subset

$$(4.53) \qquad\qquad\qquad H_\mu^* \quad =_{def} \quad \bigcap \{G(\mathcal{A}, \alpha, \epsilon, N)\}$$

with $\mathcal{A}, \alpha, \epsilon, N$ varying as in Definition 4.14.

Lemma 4.16. *If μ is a good measure on X with $S(\mu)$ $\mathbb{Q}$-like, then H_μ^* is a dense, G_δ, conjugacy invariant subset of the Polish group $H_\mu(X)$.*

Proof. Because the index set in Definition 4.14 is countable, Lemma 4.15 and the Baire Category Theorem imply that H_μ^* is a dense, G_δ subset of H_μ.

Now let $f \in H_\mu^*$, $h \in H_\mu$ and $\mathcal{A}, \alpha, \epsilon, N$ be in the index set of Definition 4.14. Let $\delta > 0$ be a positive rational modulus of uniform continuity for h and let

$$(4.54) \qquad\qquad\qquad h^{-1}\mathcal{A} \quad =_{def} \quad \{h^{-1}(A) : A \in \mathcal{A}\}$$

be the pulled back partition. Since $\mu(h^{-1}(A)) = \mu(A)$ the function α satisfies condition (4.49) for $h^{-1}\mathcal{A}$.

By assumption $f \in G(h^{-1}\mathcal{A}, \alpha, \delta, N)$. If $\mathcal{A}$ is fixed for hfh^{-1}, then $h^{-1}\mathcal{A}$ is fixed for f. Then if $\mathcal{W}$ satisfies (i)-(iv) of Definition 4.14 for f with ϵ replaced by δ, then $\{h(W_{ijsr}\}$ satisfies (i)-(iv) for hfh^{-1}. Thus, the conjugate $hfh^{-1} \in H_\mu^*$. $\qquad\square$

Now we show that when $S(\mu)$ is $\mathbb{Q}$-like, the dense G_δ subset H_μ^* is a single conjugacy class in the group $H_\mu(X)$. Thus, the group $H_\mu(X)$ satisfies the *Strong Rohlin Property*.

Theorem 4.17. *Let (Ξ, g) be the universal adding machine Cantor group with the Haar measure μ_Ξ and let $h \in H_{\mu_\Xi}(\Xi)$ be translation by the generator g. Let μ be a good measure on X with $S(\mu)$ $\mathbb{Q}$-like and let H_μ^* be the dense, G_δ subset of $H_\mu(X)$ defined by equation (4.53). If $f \in H_\mu(X)$, then $f \in H_\mu^*$ iff there exists a homeomorphism $q : X \to X \times \Xi$ such that*

$$(4.55) \qquad\qquad q_*\mu \quad = \quad \mu \times \mu_\Xi$$

and

$$(4.56) \qquad\qquad q \circ f \quad = \quad (1_X \times h) \circ q.$$

In particular, H_μ^* is a single conjugacy class in the group H_μ.

Proof. The existence of q satisfying (4.55) and (4.56) says exactly that f on X is conjugate to the fixed map $1_X \times h$ on the product $X \times \Xi$. So the set of such f in H_μ, if nonempty, consists of a single conjugacy class. If this class contains H_μ^*, then it equals H_μ^* because this set is conjugacy invariant by Lemma 4.16.

It remains to show that for a given $f \in H_\mu^*$ there exists q satifying (4.55) and (4.56).

The partitions which are fixed by f are directed by refinement. Define X_F to be the inverse limit space. So for each fixed partition $\mathcal{A}$ there is a projection map $\pi_F^{\mathcal{A}} : X_F \to \mathcal{A}$. For each $x \in X$ the carriers $\pi^{\mathcal{A}}(x)$ defined by (2.30) define a point $\pi^F(x) \in X_F$. So we obtain $\pi^F : X \to X_F$ such that

$$(4.57) \qquad\qquad \pi_F^{\mathcal{A}} \circ \pi^F \ = \ \pi^{\mathcal{A}} : X \to \mathcal{A}.$$

Since f induces the identity on each fixed partition it is clear that π^F maps f to 1_{X_F}.

Define μ_F on X_F by

$$(4.58) \qquad\qquad \mu_F \quad =_{def} \quad (\pi^F)_*\mu.$$

If $z \in X_F$, then

$$(4.59) \qquad\qquad (\pi^F)^{-1}(z) \quad = \quad \bigcap\{\pi_F^{\mathcal{A}}(z) : \mathcal{A} \text{ fixed}\}.$$

By compactness this set is not empty and so π^F is surjective and μ_F is full.

If V is a clopen subset of X_F, then there exists a fixed partition $\mathcal{A} = \{A_1, ..., A_n\}$ numbered so that for some $p \in [n]$

$$(4.60) \qquad\qquad V \quad = \quad \bigcup_{i=1}^{p} (\pi_F^{\mathcal{A}})^{-1}(A_i),$$

and so

$$(4.61) \qquad (\pi^F)^{-1}(V) \ = \ \bigcup_{i=1}^{p} A_i \qquad \text{and} \qquad \mu_F(V) = \sum_{i=1}^{p} \mu(A_i).$$

Now suppose that $\gamma \in S(\mu) \setminus \{0\}$ and $\gamma < \mu_F(V)$. Let q be the element of $[p]$ so that

$$(4.62) \qquad \sum_{i=1}^{q-1} \mu(A_i) \; < \; \gamma \; \leq \; \sum_{i=1}^{q} \mu(A_i)$$

and let

$$(4.63) \qquad \alpha_q \quad =_{def} \quad (\gamma - \sum_{i=1}^{q-1} \mu(A_i))/\mu(A_q).$$

If $\alpha_q = 1$, i.e. equality holds on the right in (4.62), then $U = \bigcup_{i=1}^{q} (\pi_F^{\mathcal{A}})^{-1}(A_i)$ is a clopen subset of V with μ_F measure γ.

Now assume that $\alpha_q < 1$ so that $\alpha_q \in S(\mu_{A_q}) \setminus \{0,1\}$. Because $f \in H_\mu^*$ it is in $G(\mathcal{A}, \alpha, 1, 1)$ with α any map satisfying (4.49) but with α_q given by (4.63). Let $\mathcal{W}$ be a partition which satisfies (i)-(iv) of Definition 4.14 and let

$$(4.64) \qquad \begin{aligned} A_q^- \quad &=_{def} \quad \bigcup \{W_{q1sr} : s \in [M] \text{ and } r \in [k]\}, \\ A_q^+ \quad &=_{def} \quad \bigcup \{W_{q1sr} : s \in [M] \text{ and } r \in [k]\} \; = \; A_q \setminus A_q^-. \end{aligned}$$

By condition (iv), $\tilde{\mathcal{A}} = \{A_1, ..., A_{q-1}, A_q^-, A_q^+, A_{q+1}, ..., A_n\}$ is a fixed partition refining $\mathcal{A}$ and by (iii)

$$(4.65) \qquad \mu(A_q^-) \; = \; \alpha_q \mu(A_q) \; = \; \gamma - \sum_{i=1}^{q-1} \mu(A_i).$$

It follows that $U = (\bigcup_{i=1}^{q-1} (\pi_F^{\tilde{\mathcal{A}}})^{-1}(A_i)) \cup (\pi_F^{\tilde{\mathcal{A}}})^{-1}(A_q^-)$ is a clopen subset of V with μ_F measure γ.

First, this shows that X_F has no isolated points and so, as the countable inverse limit of finite sets, it is a Cantor space. Next, since γ can be chosen from elements of a dense subset of $(0, \mu_F(V))$ it follows that the measure μ_F is nonatomic. Thus, $\mu_F \in \mathcal{M}_{X_F}$. Then by applying the argument to $V = X_F$ we see that $S(\mu) \subset S(\mu_F)$. The reverse inclusion follows from Proposition 1.5 and so

$$(4.66) \qquad S(\mu_F) \quad = \quad S(\mu).$$

Finally, the argument for general clopen V is a direct verification of the Subset Condition and so μ_F is a good measure on X_F.

For the adding machine factor an inductive construction with repeated choices is required because the cyclic partitions are *not* directed by refinement.

On the finite pointed cyclic group $(\mathbb{Z}_k, 1)$ of integers mod k, let μ_k be the Haar measure, i.e. the uniform distribution, and let h_k be the 1 translation map.

If $\mathcal{A}$ is an invariant partition for f, then a $\mathbb{Z}_k$ *label* for $\mathcal{A}$ is a map $u_k : \mathcal{A} \to \mathbb{Z}_k$ such that

$$(4.67) \qquad u_k \circ f_{\mathcal{A}} \quad = \quad h_k \circ u_k.$$

That is, u_k maps $f_{\mathcal{A}}$ to h_k. Since μ_k is the only h_k invariant measure, (4.27) and (4.67) imply

$$(4.68) \qquad u_{k*} \mu_{\mathcal{A}} \quad = \quad \mu_k;$$

see Proposition 4.4(d). With $[\mathcal{A}]$ the fixed partition associated with $\mathcal{A}$ obtained by concatenating the elements of each cycle of $f_{\mathcal{A}}$ we have

$$
\begin{aligned}
[u_k] \quad &=_{def} \quad \pi_{\mathcal{A}}^{[\mathcal{A}]} \times u_k : \mathcal{A} \to [\mathcal{A}] \times \mathbb{Z}_k, \\
[u_k] \circ f_{\mathcal{A}} \quad &= \quad 1_{[\mathcal{A}]} \times h_k \circ [u_k], \\
[u_k]_* \mu_{\mathcal{A}} \quad &= \quad \mu_{[\mathcal{A}]} \times \mu_k
\end{aligned}
$$
(4.69)

because the only $1_{[\mathcal{A}]} \times h_k$ invariant measure which projects to $\mu_{[\mathcal{A}]}$ is the product measure.

Of course, an invariant partition $\mathcal{A}$ admits a $\mathbb{Z}_k$ label iff every cycle in the permutation $f_{\mathcal{A}}$ has length divisible by k. Given the label, $[u_k]$ is surjective and it is bijective iff every $f_{\mathcal{A}}$ cycle has length exactly k.

Recall that $f \in H_\mu^*$. Hence, for every positive rational ϵ and positive integer N there exists a positive integer k and an invariant partition $\mathcal{W}(\epsilon, kN)$ whose mesh is less than ϵ and such that every $f_{\mathcal{W}}$ cycle has length kN. Hence, there is a $\mathbb{Z}_{kN}$ label u_{kN} on $\mathcal{W}$ such that $[u_{kN}]$ is a bijection from $\mathcal{W}$ to $[\mathcal{W}] \times \mathbb{Z}_{kN}$. In fact, with the indexing given by (i)-(iv) of Definition 4.14 the r index provides the map u_{kN} and ijs indexes the elements of $[\mathcal{W}]$.

We will need one other construction.

Suppose that $k_1 | k_2$. The natural projection $\pi_{k_2}^{k_1} : (\mathbb{Z}_{k_2}, 1) \to (\mathbb{Z}_{k_1}, 1)$ maps h_{k_2} to h_{k_1}. On the product group $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$ the product map $h_{k_1} \times h_{k_2}$ is translation by the element $(1, 1)$ and every orbit is periodic with period k_2.

These groups are in fact rings and $\pi_{k_2}^{k_1}$ is a ring homomorphism. So we can regard the product $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$ as a module over $\mathbb{Z}_{k_2}$ and so can multiply by elements of $\mathbb{Z}_{k_2}$.

An element $(i, j) \in \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$ can be written uniquely as

$$
(4.70) \qquad (i, j) = (k, 0) + j(1, 1) \qquad \text{with} \quad k = i - \pi_{k_2}^{k_1}(j).
$$

Now choose a set map $\tilde{v} : \mathbb{Z}_{k_1} \to \mathbb{Z}_{k_2}$ which is a right inverse for the surjection $\pi_{k_2}^{k_1}$. That is, $\pi_{k_2}^{k_1} \circ \tilde{v}$ is the identity on $\mathbb{Z}_{k_1}$. Now extend $\tilde{v}$ to map $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$ onto $\mathbb{Z}_{k_1}$ so that

$$
(4.71) \qquad (i, j) = (k, 0) + j(1, 1) \qquad \Longrightarrow \qquad v(i, j) = \tilde{v}(k) + j.
$$

That is, from (4.70)

$$
(4.72) \qquad\qquad v(i, j) = \tilde{v}(i - \pi_{k_2}^{k_1}(j)) + j.
$$

Since $\pi_{k_2}^{k_1}$ is a homomorphism, it follows that

$$
(4.73) \qquad\qquad \pi_{k_2}^{k_1} \circ v(i, j) = i.
$$

Furthermore,

$$
\begin{aligned}
v((i, j) + j_1(1, 1)) &= v(i + \pi_{k_2}^{k_1}(j_1), j + j_1) \\
&= \tilde{v}(i + \pi_{k_2}^{k_1}(j_1) - \pi_{k_2}^{k_1}(j + j_1)) + j + j_1 \\
&= \tilde{v}(i - \pi_{k_2}^{k_1}(j)) + j + j_1 = v(i, j) + j_1.
\end{aligned}
$$
(4.74)

That is, v maps $h_{k_1} \times h_{k_2}$ to h_{k_2}.

For our inductive construction, let $\{\epsilon_n\}$ be a sequence of positive rationals tending to 0 with $\epsilon_0 > d(X)$. Define $\mathcal{A}_0 = \{X\}, k_0 = 1, u_1 : \mathcal{A}_0 \to \mathbb{Z}_1$ mapping X to 1.

Inductively, suppose that $\mathcal{A}_n$ is an invariant partition with mesh less than ϵ_n and for some positive integer k_n with $n|k_n$ there is a $\mathbb{Z}_{k_n}$ label u_{k_n} on $\mathcal{A}_n$ such that $[u_{k_n}]$ is a bijection from $\mathcal{A}_n$ to $[\mathbb{Z}_1] \times \mathbb{Z}_{k_n}$, i.e. every $f_{\mathcal{A}_n}$ cycle has length k_n.

Because $f \in H_\mu^*$ there exists a positive integer k_{n+1} such that

(4.75) $$(n+1)k_n|k_{n+1}$$

and an invariant partition $\mathcal{W}(\epsilon_{n+1}, k_{n+1})$ such that every $f_{\mathcal{W}}$ cycle has length k_{n+1}. Choose $\tilde{u} : \mathcal{W} \to \mathbb{Z}_{k_{n+1}}$ a $\mathbb{Z}_{k_{n+1}}$ label.

Define $\mathcal{A}_{n+1} = \mathcal{W} \wedge \mathcal{A}_n$. So $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$ and since $k_n|k_{n+1}$ every $f_{\mathcal{A}_{n+1}}$ cycle has length k_{n+1}. We obtain a $\mathbb{Z}_{k_{n+1}}$ label, $u_{k_{n+1}} : \mathcal{A}_{n+1} \to \mathbb{Z}_{k_{n+1}}$ by composing the maps

(4.76) $$(u_{k_n} \circ \pi_{\mathcal{A}_{n+1}}^{\mathcal{A}_n}) \times (\tilde{u} \circ \pi_{\mathcal{A}_{n+1}}^{\mathcal{W}}) : \mathcal{A}_{n+1} \to \mathbb{Z}_{k_n} \times \mathbb{Z}_{k_{n+1}}$$
$$\text{and} \quad v : \mathbb{Z}_{k_n} \times \mathbb{Z}_{k_{n+1}} \to \mathbb{Z}_{k_{n+1}}.$$

Thus, we have constructed $\mathcal{A}_{n+1} \to \mathcal{A}_n$ of mesh less than ϵ_{n+1} with a $\mathbb{Z}_{k_{n+1}}$ label $u_{k_{n+1}}$ such that $n+1|k_{n+1}$ and $k_n|k_{n+1}$ and with $[u_{k_{n+1}}]$ a bijection. In addition, (4.73) implies that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{A}_{n+1} & \xrightarrow{[u_{k_{n+1}}]} & [\mathcal{A}_{n+1}] \times \mathbb{Z}_{k_{n+1}} \\
\pi_{\mathcal{A}_{n+1}}^{\mathcal{A}_n} \downarrow & & \downarrow \pi_{[\mathcal{A}_{n+1}]}^{[\mathcal{A}_n]} \times \pi_{k_{n+1}}^{k_n} \\
\mathcal{A}_n & \xrightarrow[{[u_{k_n}]}]{} & [\mathcal{A}_n] \times \mathbb{Z}_{k_n}
\end{array}
$$

Now we identify the inverse limit spaces of these three sequences, together with their maps and measures. Since each open set in the inverse limit is a pullback of an open set from some factor, it follows that the measure on the inverse limit is uniquely determined by the projections to the factors.

For every positive integer m we have $m|k_n$ if $n > m$. Hence, the adding machine group which is the inverse limit of the pointed cyclic groups $\{(\mathbb{Z}_{k_n}, 1)\}$ is the universal adding machine (Ξ, g) and the inverse limit of the translation maps h_{k_n} is the g translation map h. The Haar measure μ_Ξ is the measure which projects to the uniform measures on the factors.

The map which associates to $x \in X$ the sequence of carriers $\pi^{\mathcal{A}_n}(x)$ is a surjection by compactness and it is injective because the sequence of meshes $\{mesh(\mathcal{A}_n)\}$ tends to zero. So we obtain a homeomorphism which identifies X with the inverse limit of the sequence $\{\mathcal{A}_n\}$. The map f on X projects to $f^{\mathcal{A}_n}$ and so f is identified with the inverse limit of these maps. The measure μ is the unique measure on X which projects to each $\mu_{\mathcal{A}_n}$.

If $\mathcal{A}$ is any partition on X, then it has a positive Lebesgue number $l(\mathcal{A})$. When n is large enough that $mesh(\mathcal{A}_n) < l(\mathcal{A})$, then $\mathcal{A}_n$ refines $\mathcal{A}$. If $\mathcal{A}$ is fixed, then it follows that $[\mathcal{A}_n]$ refines $\mathcal{A}$. Thus, the sequence $\{[\mathcal{A}_n]\}$ is cofinal in the directed system of fixed partitions. Hence, the inverse limit of the sequence $\{[\mathcal{A}_n]\}$ is X_F and the measure μ_F is the one with the correct projections.

From the above commutative diagram we see that on the inverse limits the sequence of bijections $\{[u_{k_n}]\}$ induces a homeomorphism $[u] : X \to X_F \times \Xi$. By (4.69) it maps f on X to the identity on X_F and the translation map h on Ξ and the measure μ on X to μ_F on X_F and μ_Ξ on Ξ.

Finally, μ_F is a good measure on the Cantor space X_F and so (4.66) and Theorem 2.9 imply that there is a homeomorphism $\tilde{q} : X_F \to X$ which maps μ_F to μ. The required map $q = (\tilde{q} \times 1_\Xi) \circ [u] : X \to X \times \Xi$. $\square$

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